Chapter 2: Resolving indeterminate forms — **Section 1**

Indeterminate forms

and some theoretical tools about them

What you need to know already:	What you can learn here:
➤ The concepts of limit and continuity.	➤ How to approach limit situations that are not clear on a first glance.

So far we have seen how to determine limits when they are *boring* – although they are irrelevant there – and when those simple "*laws*" involving infinity can be used. But that leaves out the most interesting situations, namely, those situations where something interesting occurs in the formula of the function, but we cannot see immediately what the limit turns out to be, or even if there is one!

We know that both graphical and numerical approaches provide only estimates, but no guarantees. In this chapter we'll develop ***algebraic*** methods that often (but not always) provide the exact value of a limit. These will apply to what are generally called indeterminate forms, but before I give you a specific definition of what this means, I need to point out an important suggestion.

Knot on your finger

Even though a limit explains the behaviour of the function ***near*** a certain value, the ***first step*** in computing an unclear limit should always be to determine what form the function takes ***at*** that value.

Such check will go a long way towards helping us decide which method is more likely to work.

Example: $\lim_{x\to 3}\dfrac{x^2-9}{x^3-27}$

In this case the function is not defined at the value of interest, but we can try to evaluate it there anyway. By doing so, we get 0 both on top and bottom, that is, we get the form $\dfrac{0}{0}$, which is meaningless in itself, but, as we shall see soon, tells us how we can go about computing the limit.

So, by trying to evaluate the function, we get something that we cannot compute, but tells us what to do next, right?

Absolutely. And that is why we call these forms: they are not computable quantities, but their structure give us hints on how to proceed. And with this in mind, here are the indeterminate forms that we shall consider.

Definition

A function $y = f(x)$ is said to have an ***indeterminate form*** at $x = c$ (where *c* can be finite or infinite) if:

1. $f(x)$ is ***continuous*** on an interval including *c*, except possibly at *c*.
2. When we try to evaluate $f(x)$ at *c* we obtain ***one of the following forms***:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\times\infty, \quad \infty-\infty, \quad 1^{\infty}, \quad \infty^{0}, \quad 0^{0}$$

Here 0 and 1 represent ***variable*** quantities approaching the respective value, ***NOT*** constants with that value.

When ***resolving*** an indeterminate form, we use appropriate methods to determine if the limit exists and, if so, what its value is.

But if any number times 0 is 0, why is $0\times\infty$ indeterminate?

Notice the last sentence in the definition: in these forms we are not dealing with the number 0, but with some variable quantity that is approaching 0. That makes all the difference. To clarify what we are dealing with, let me explain why each of these forms is indeterminate.

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The form $\frac{0}{0}$ is indeterminate because it represents the limit of the ratio of ***two very small quantities***. This ratio can approach any limit, depending on the relative size of the two quantities.

Example:

In $\lim_{x\to 0}\frac{x^2}{x^4}$ both numerator and denominator approach 0, so this is a $\frac{0}{0}$ form. T resolve this form, we notice that as long as we stay away from 0, this limit is the same as $\lim_{x\to 0}\frac{1}{x^2}$, and the law of balloons tells us that this limit is therefore ∞.

If we look at $\lim_{x\to 0}\frac{x^4}{x^2}$, we still get a $\frac{0}{0}$ form, but this time away from 0 this is the same as $\lim_{x\to 0}x^2$, which tells us that the limit is 0.

Finally, $\lim_{x\to 0}\frac{ax^2}{x^2}$ also provides a $\frac{0}{0}$ form, but it obviously approaches *a*, whatever number *a* is.

Therefore, a $\frac{0}{0}$ form is indeterminate in that it can lead to any limit, and even not have a limit!

Example:

As in the previous example, just compare the following three limits and notice that they all provide a $\frac{0}{0}$ form, and yet have different limiting values:

$$\lim_{x\to3}\frac{x-3}{x^2-9} \quad ; \quad \lim_{x\to3}\frac{(x-3)^2}{x^2-9} \quad ; \quad \lim_{x\to3}\frac{x-3}{(x^2-9)^2}$$

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The form $0\times\infty$ is indeterminate because it represents the limit of the product of a ***very large and a very small quantity***. This product can approach any limit, depending on the relative size of the two quantities.

Example:

Use the functions of the previous two examples, but think of them as the product of the numerator and the reciprocal of the denominator:

$$\lim_{x\to0}\frac{x^2}{x^4}=\lim_{x\to0}x^2\times\frac{1}{x^4} \quad ; \quad \lim_{x\to0}\frac{x^4}{x^2}=\lim_{x\to0}x^4\times\frac{1}{x^2}$$

$$\lim_{x\to3}\frac{x-3}{x^2-9}=\lim_{x\to3}(x-3)\frac{1}{x^2-9}$$

and so on. If we try to evaluate them, we always get a $0\times\infty$ form and yet, as we have seen, these forms lead to different limits, hence they are indeterminate.

The next two cases follow the same rationale as the previous ones, so I will delay examples to after we have seen some more methods to resolve indeterminate forms.

Knot on your finger

The form $\frac{\infty}{\infty}$ is indeterminate because it represents the limit of the ratio of ***two very large quantities***. This ratio can approach any limit, depending on the relative size of the two quantities.

Knot on your finger

The form $\infty-\infty$ is indeterminate because it represents the limit of the difference between ***two very large*** quantities. This difference can approach any limit, depending on the relative size of the two quantities.

The remaining forms involve powers and can be deceiving as they may remind you of certain familiar facts about power, except that those facts do not work in limit situations!

Unfortunately I will have to delay examples on these cases too, until we look at more methods to compute limits. For now just focus on the reasons given and bring to your instructor any doubts or puzzlements you may have.

Knot on your finger

The form 1^{∞} is indeterminate because it represents the limit obtained by raising a quantity ***close*** to 1 to a large exponent. Since larger powers of a number greater than 1 grow, while larger power of a quantity smaller than 1 get smaller, this is indeterminate.

You may think that raising 1 to any power always gives 1, which is true, but here we are not raising 1 to a power, but a number *close* to 1. The relative size of base and power may lead to different limits

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The form ∞^{0} is indeterminate because it represents the limit obtained by raising a ***large quantity to a small exponent***. Since a small exponent makes the power of a large quantity smaller, base and exponent tend to opposite directions, so that this form can take any value, depending on the relative size of base and exponent.

This time you may think that any number to the 0 power is 1, but here we are not raising that number to 0, but to an exponent *close* to 0. That makes all the difference and makes the form indeterminate.

Knot on your finger

The form 0^{0} is indeterminate because it represents the limit obtained by raising a ***small quantity to a small exponent***. Since a small exponent makes the power of a small quantity larger, base and exponent tend to opposite directions, so that this form can take any value, depending on the relative size of base and exponent.

For the last indeterminate form, notice also that $x^{0} = 1$, but $0^{x} = 0$ for any small positive number *x*, so would 0^{0} tend to 0 or 1? You can see that we are dealing with an indeterminate form.

I hope I have generated enough curiosity about these forms that you will be encouraged to learn the methods discussed in the next sections and thus see some really interesting limits for indeterminate forms. And since I am at it, let me finish with another classical fact about limits that will be used later (and only once, really!) but has a very convincing graphical rationale.

Technical fact:

the Squeeze or Sandwich Theorem

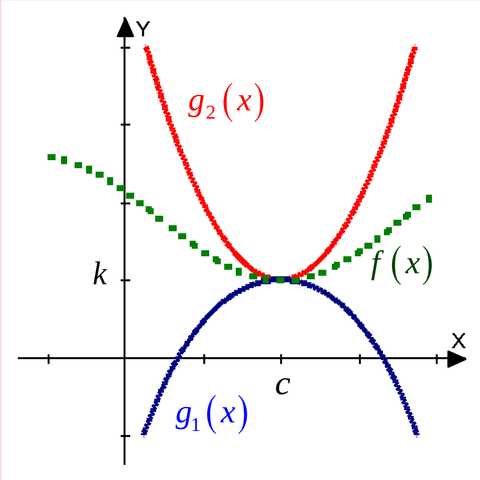
If $f(x), g_1(x), g_2(x)$ are functions and c is a value such that:

- $f(x), g_1(x), g_2(x)$ are continuous on an interval around c, except possibly at c
- $g_1(x) \le f(x) \le g_2(x)$ in such interval
- $\lim\limits_{x \to c} g_1(x) = \lim\limits_{x \to c} g_2(x) = k$

then $\lim\limits_{x \to c} f(x) = k$. In less formal terms, if $f(x)$ is squeezed, or sandwiched, between two functions that go to the same limit, it goes to the same limit.

Proof

Look at the picture: what else can possibly happen? OK, this is a very informal argument, but it is true and backed up by formal arguments that mathematicians have checked, so you can trust it ☺.



As you can tell, this was mostly a conceptual section, so the *Learning questions* will reflect such emphasis.

Summary

- An indeterminate form occurs when the function would take on an expression that cannot be computed and for which the basic limit laws do not provide a definitive value.
- In that case we need other ways to compute the limit so as to determine what value the function is approaching.
- It is important to distinguish indeterminate forms, which must be resolved through appropriate and more advanced limit methods, from other non-numerical forms that can be evaluated by using basic limit laws.

Common errors to avoid

- Do not apply familiar properties of powers and rules of algebra to indeterminate forms!

Learning questions for Section 2.1

Memory questions:

1. What is the first step when analyzing an indeterminate limit form?

2. What are the four indeterminate forms involving arithmetic operations?

3. What are the three indeterminate forms involving powers?

Computation questions:

For the functions of questions 1-20:

a) Identify the values of x for which an indeterminate form occurs
b) Determine the type of indeterminate form it is.
c) Explain why it is indeterminate.
d) If you know of a method to resolve this indeterminate form, do so and compute the limit.

1. $y=\dfrac{t^3+8}{t+2}$

2. $y=\dfrac{5x^2}{3x^2-x}$

3. $y=\dfrac{s^3-1}{s^2-1}$

4. $y=\dfrac{1}{\sqrt{4-x^2}}$

5. $y=\dfrac{x+2x^2}{x}$

6. $y=\dfrac{x}{e^x-1}$

7. $y=\dfrac{\ln\left(x^2-1\right)}{x^2-4}$

8. $y=\dfrac{\sqrt{4-z}-\sqrt{2}}{z^2-4}$

9. $y=\left(1+x\right)^{1/x}$

10. $y=\dfrac{x-3}{\sqrt{x^2-6x+9}}$

11. $y=\dfrac{1}{t-2}-\dfrac{4}{t^2-4}$

12. $y=\dfrac{x\sin x}{1-\cos x}$

13. $y=\dfrac{\sin x}{\sqrt{x}}$

14. $y=\dfrac{\tan 7x}{\sin 3x}$

15. $y=\dfrac{\cos 2x+1}{x-\frac{\pi}{2}}$

16. $y = \begin{cases} \dfrac{3x-5}{x^2-1} & \text{if } x<0 \\ \tan x & \text{if } 0 \le x \le 2 \\ \dfrac{x^2-2x-3}{x^2-3x} & \text{if } x>2 \end{cases}$

17. $y = \begin{cases} \dfrac{2x^3+x^2+1}{x^2-1} & \text{if } x<1 \\ \sqrt[3]{(x+7)^2} & \text{if } 1 \le x \le 2 \\ \dfrac{x^2-1}{x^2-9} & \text{if } x>2 \end{cases}$

18. $y = \ln\dfrac{2}{x+2}$

19. $y = e^{3x}$

20. $y = e^{3/x}$

21. If $y = f(x)$ is a function such that $3-e^{-x} < f(x) < 4-\tanh x$, what is its limit as x approaches infinity?

22. If a function $f(x)$ is such that for any $x \ge 5$, $\dfrac{2x^2-\cos x}{3-x^2} < f(x) < -1-e^{\frac{3-x}{x^2}}$, what is its limit at infinity?

Theory questions:

1. Explain why 0^0 and ∞^0 are not equal to 1.

2. Provide a valid argument explaining why each of the following forms is not indeterminate:

   a) $\dfrac{0}{\infty}$ b) 0^{∞} c) $\infty^{-\infty}$ d) $\dfrac{\infty}{0}$ e) ∞^1

3. In what sense can the form #/0 be considered indeterminate and why was it not listed as one of the indeterminate forms in this section?

Templated questions:

1. For any function you see, identify the values that may generate an indeterminate form, check if it is indeterminate and, if so, of what kind.

What questions do you have for your instructor?