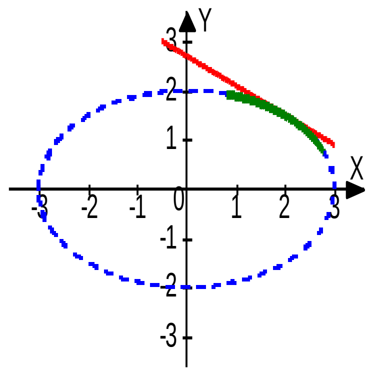
Chapter 4: Basic differentiation rules — **Section 5**

Implicit differentiation

What you need to know already:	What you can learn here:
➤ Basic rules of differentiation, including the chain rule.	➤ The method of implicit differentiation, useful to compute slopes on curves that are not functions.

The concept of derivative was developed to compute the slope of the tangent line to a function. However, there are curves that are not functions, but still have a tangent line. Can we use differentiation rules to compute the slope of their tangent lines, even though we are not dealing with a function?
The answer is ***yes***, as long as the curve represents a function on a restricted domain and range.

Example: $4x^2 + 9y^2 = 36$



The graph of this equation is the *ellipse* shown here. Clearly, this is not a function, but the line you see is its geometric tangent line, an observation that suggests that the concept of derivative should apply here in some useful and reasonable way.

Well, notice that if we focus only on the portion of the ellipse that is close to that tangent line, and ignore the rest, we get a curve which is indeed the graph of a function, since it passes the vertical line test. This theoretical restriction can be done for all points on the ellipse, except for left or right vertices.

So, at least theoretically, we should be able to compute the slope of the tangent line by computing derivatives. But how do we do that if we do not have an explicit formula for the function?

One way is to solve the equation of the curve for *y*, thus computing the needed derivative in the usual way. This can be done, for instance, in the previous example, but it is usually a long and/or difficult process and sometimes it is just impossible.

But there is a better, simpler and faster way to achieve the same goal.

Strategy for finding the slope of an implicitly defined curve

If a curve is described ***implicitly***, that is, by means of an equation in *x* and *y*, the slope of a line tangent to it can be obtained by:

1. ***differentiating*** left and right sides of the equation,
2. treating *y* as an unknown, ***implicit*** function of *x*,

3. using the ***chain rule*** whenever *y* is composed inside another function, and
4. ***solving*** for *y*' after the differentiation is done.

Ah! Implicit differentiation because we think of y as an implicit, unknown function of x!

Yes, and also because we compute the derivative without even knowing what the original function is.

Example: $4x^2+9y^2=36$

We can compute the slopes of the lines tangent to this curve by using implicit differentiation.

We differentiate both sides:

$$\frac{d}{dx}\left(4x^2+9y^2\right)=\frac{d}{dx}(36)$$

Now we apply the appropriate rules, including the chain rule, when we differentiate the term that contains *y*:

$$\frac{d}{dx}\left(4x^2\right)+\frac{d}{dx}\left(9y^2\right)=0 \quad\Rightarrow\quad 8x+9\frac{d}{dx}\left(y^2\right)=0$$

$$\Rightarrow\quad 8x+9\frac{d}{dy}\left(y^2\right)\frac{dy}{dx}=0 \quad\Rightarrow\quad 8x+18y\frac{dy}{dx}=0$$

Finally we solve for *y*':

$$\Rightarrow\quad \frac{dy}{dx}=-\frac{8x}{18y}=-\frac{4x}{9y}$$

By using suitable coordinates $(x,\ y)$ for the points on the curve, we can obtain the required slopes.

I will give you just one more example for now, but we shall see plenty more once we learn how to differentiate more transcendental functions and how to solve applied problems that require this method.

Example: $\sqrt{xy}=x-2y$

Solving this equation for *y* may be a (not so) nice way to spend a rainy evening, but if all we need is a formula for the derivative, we can use the faster method of implicit differentiation. In fact, since we are going to differentiate both sides and all we need is a relation between the two variables, we can simplify our work further by squaring both sides, so as to avoid differentiating the square root:

$$\sqrt{xy}=x-2y \quad\Rightarrow\quad xy=x^2-4xy+4y^2$$

Next we differentiate both sides, by using the chain rule for terms involving *y*, as well as any other appropriate rule:

$$\Rightarrow\quad y+xy'=2x-4y-4xy'+8yy'$$

Finally we solve for *y*':

$$\Rightarrow\quad 5xy'-8yy'=2x-5y \quad\Rightarrow\quad y'=\frac{2x-5y}{5x-8y}$$

How difficult will the last step of solving for y' be? With this method we may end up with complicated equations.

We may indeed end up with complicated equations involving *x*, *y* and *y*', but…

Knot on your finger

The method of implicit differentiation produces ***always*** a ***linear equation*** in y'. Therefore, such equation can always be solved by simply isolating the needed y'.

Summary

- When a curve is represented by an implicit relation that links the two variables through a generic equation, implicit differentiation can be used to compute the slope of the curve without having to solve the equation explicitly for the dependent variable.
- The method of implicit differentiation is based on the use of the chain rule.
- The method of implicit differentiation has many important applications, both in calculus theory and in applied problems.

Common errors to avoid

- Remember to apply the chain rule whenever the dependent variable appears itself within a function in the equation. Confusing? That is why I am pointing this out as an error to avoid!

Learning questions for Section D 4-5

Review questions:

1. Describe when and how implicit differentiation is used.
2. Explain why implicit differentiation is called in this way

Memory questions:

1. Which differentiation rule is always used in implicit differentiation?
2. The slope of what type of curves can be found by using implicit differentiation?

Computation questions:

For each of the curves whose equation is presented in questions 1-22:

a) Find *dy/dx* by implicit differentiation
b) Determine the equation of the line tangent to the given curve at the given point on it.
c) If possible, find *dy/dx* by solving explicitly for *y* first, then compare your answer to what you obtained in a).

1. $7y^2 - xy^3 = 4$ at $(3, 2)$
2. $x^2 + 4y^2 = 9$ at $(1, \sqrt{2})$
3. $x^{4/3} + y^{4/3} = 17$ at $(1, 8)$.
4. $x^{1/3} - y^{2/3} = 1$ at $(8, -1)$.
5. $y^2 - 3xy + x^3 = e^{\pi}$ at $(0, e^{\pi/2})$
6. $y^3 + x^3 = 8$ at $(2, 0)$
7. $(x^2 + y^2)^2 = 4x^2 y$ at $(1, 1)$
8. $x^3 + 3y^2 = 3xy + 7x$ at $(3, 1)$
9. $\sqrt{x + y^2} = 1 + x^2 y$ at $(0, 1)$
10. $x^2 + 2y^2 = 4xy + 2$ at the *y* intercept.
11. $x^3 y^3 + 3xy = 4x^2 y^2$ at $(1, 1)$
12. $x^3 y^2 - 2x^2 y = -xy^2$ at $(1, 1)$
13. $x^{3/4} + y^{3/4} = 9$ at $(1, 16)$
14. $\sqrt{xy} = x - 2y$ at the origin.
15. $y^2 = \sqrt{3 - 2x}$ at $(1, -1)$
16. $y^2 = \sqrt{3 - 2x}$ at $(1, 1)$
17. $\dfrac{4y - 6xy^2}{x^2 - 2y^2} = 1$ at $(0, -2)$.
18. $x^3 - xy^4 - x^2 y = 2$ at $(2, 1)$.
19. $y^2 = x^3 - 2x + 4$ at $(3, -5)$. This is called an *elliptic curve*, even though its graph is not an ellipse.
20. $y^2 = x^3 - 7x + 6$ at $(3, -\sqrt{12})$. This is also called an *elliptic curve*, even though its graph is not an ellipse.
21. $3(x^2 + y^2)^2 = 100xy$ at $(3, 1)$. This is called a *lemniscate*.
22. $(4 - x)y^2 = x^3$ at $(2, 2)$. This is called a *cissoid*.

For each of the functions presented in questions 23-24, compute the derivative by using first appropriate differentiation rules and then by applying implicit differentiation to the equation obtained by eliminating the root. Check that the conclusion is the same with both methods.

23. $f(x)=\sqrt{3-2x}$

24. $g(x)=\sqrt[3]{3x-x^2}$

Proof questions:

1. Use implicit differentiation to prove that the power rule works for $y=\sqrt{x}$.

2. Use implicit differentiation to prove that the power rule works for $y=\sqrt[n]{x^m}$, where *m* and *n* are two positive integers and *n*>1.

Application questions:

1. At which of the points on the graph of the equation $x^2y+xy^2=16$ is the tangent line horizontal?

2. At which of the points on the graph of the equation $x^2y+xy^2=9$ is the tangent line vertical?

3. At which of the points on the graph of the equation $x^3y^2+x^2y^3=9$, if any, is the tangent line vertical?

Templated questions:

1. Construct a simple equation in *x* and *y* that is not easily solved for *y*, then compute *dy/dx*.

What questions do you have for your instructor?