Chapter 4: Basic differentiation rules — **Section 6**

Derivatives of inverse functions

What you need to know already:	What you can learn here:
➤ The basic rules of differentiation and implicit differentiation.	➤ How to use implicit differentiation to construct rules for inverse functions.

Although we looked at the method of implicit differentiation to compute the slopes for curves that are *not* functions, its first extended application is that of computing the derivative of a whole set of functions, namely inverse functions. Here is how this simple method works.

Strategy to obtain the derivative of an inverse function

If $f(x)$ is an invertible function whose derivative $f'(x)$ is known, to compute the derivative of the inverse:

1. **Express** y as the inverse function: $y = f^{-1}(x)$
2. **Apply the function** $f(x)$ to both sides, thus obtaining $f(y) = x$
3. **Differentiate** both sides implicitly:
$$f'(y) \times y' = 1$$
4. **Solve** for y': $y' = \dfrac{1}{f'(y)}$
5. **Express** $f'(y)$ in terms of x by using the original relation between x and y.

Example: $y = \sqrt{x}$

When I presented the power rule, I stated that it works for any value of the exponent, but so far I have only given you the proof for positive and negative integer exponents. But does it work for fractional exponents? This gives us the first opportunity to use our method. If all goes well, the derivative of $y = \sqrt{x} = x^{1/2}$ should be $y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. Let's see…

The strategy I just outlined applies here since the square root is the inverse of the square function, for which we know that the power rule works. So:

1. Start from our function: $y = \sqrt{x}$
2. Apply its inverse to both sides: $y^2 = x$

3. Differentiate both sides implicitly: $2y \times y' = 1 \quad \Rightarrow \quad y' = \dfrac{1}{2y}$

4. Since we know that $y = \sqrt{x}$, we conclude that: $y' = \dfrac{1}{2\sqrt{x}}$, as we expected.

Knots on your finger

If you are clear on how the procedure works, when you apply it, you may be able to jump directly to the fourth step.

The fifth step may require the use of special identities, which in some situations will create difficulties.

Can we use this method to compute the derivative of the natural logarithm, since we know the derivative of its inverse, the natural exponential?

Yes, and, not surprisingly, it turns out to be as simple as the derivative of the exponential!

Technical fact:

The natural logarithm rule

If $y = \ln x$, then $y' = \dfrac{1}{x}$.

Proof

By using the method of implicit differentiation, we start from our function and apply its inverse, the exponential, to both sides:

$$y = \ln x \quad \Rightarrow \quad e^y = x$$

Next we differentiate implicitly both sides and isolate y':

$$\Rightarrow \quad e^y y' = 1 \quad \Rightarrow \quad y' = \frac{1}{e^y}$$

But we know that $e^y = x$, so we can conclude that:

$$(\ln x)' = \frac{1}{x}$$

as claimed.

Example: $y = \ln^2\left(x^2 + 3x\right)$

This may look complicated, but we only need the rules we have seen so far. We start with the chain rule:

$$y' = 2\ln\left(x^2+3x\right)\frac{d}{dx}\left(\ln\left(x^2+3x\right)\right)$$

Then we use the logarithm rule and the chain rule again:

$$y' = 2\ln\left(x^2+3x\right)\frac{1}{x^2+3x}\frac{d}{dx}\left(x^2+3x\right)$$

Finally, some basic rules to finish the job:

$$y' = 2\ln\left(x^2+3x\right)\frac{1}{x^2+3x}(2x+3) = \frac{4x+6}{x^2+3x}\ln\left(x^2+3x\right)$$

Once we learn how to compute derivatives of other transcendental functions, we shall use this method to find the derivative of their inverses as well. For now, just practice with simple inverses.

Summary

- The method of implicit differentiation can be used, at least theoretically, to obtain the derivative formula for the inverse of any function.
- In particular, it allows us to confirm the power rule for fractional exponents and to construct the formula for the derivative of the natural logarithm.
- Finding an explicit formula with this method may create difficulties when not enough simple identities are available.

Common errors to avoid

- Watch out when you return to the original independent variable: don't forget to do it, and do it right!

Learning questions for Section D 4-6

Review questions:

1. Explain why the method of implicit differentiation is useful to compute derivatives of inverse functions.
2. Describe in your words, but accurately, how to use the method of implicit differentiation to compute the derivative of the inverse of a function.

Memory questions:

1. To what do we change the equation $y = f^{-1}(x)$ in order to compute the derivative of an inverse function?
2. The derivative of what type of functions can be found by using implicit differentiation?
3. What is the derivative of $y = \ln x$?

Computation questions:

For each of the functions presented in questions 1-4, compute the derivative by using first appropriate differentiation rules and then by applying implicit differentiation to the equation obtained by eliminating the root. Check that the conclusion is the same with both methods.

1. $y=\sqrt{x^2+3x}$
2. $y=\sqrt[3]{e^x-x}$
3. $f(x)=\sqrt{3-2x}$
4. $g(x)=\sqrt[3]{3x-x^2}$

Compute the derivative of each of the functions presented in questions 5-6.

5. $y=\ln\left(4x-\dfrac{1}{x}\right)$
6. $y=\ln\left(e^{3x}-\sqrt{x+4}\right)$

Theory questions:

1. Technically speaking, the fact that $(\ln x)'=\dfrac{1}{x}$ only works if we add a restriction on the derivative: which one?

Proof questions:

1. Show that the power rule works for $y=x^{n/m}$, for any natural numbers *n* and *m*, by using the method of implicit differentiation on this inverse function.
2. Obtain the differentiation rule for a general logarithm $y=\log_a x, a>0$.
3. Prove that $\dfrac{d}{dx}\left(\ln|x|\right)=\dfrac{1}{x}$.

What questions do you have for your instructor?