**Roberto’s Notes on Differential Calculus**

**Chapter 5: Derivatives of transcendental functions**

**Section 3**

**Derivatives of Hyperbolic functions**

<table>
<thead>
<tr>
<th>What you need to know already:</th>
<th>What you can learn here:</th>
</tr>
</thead>
<tbody>
<tr>
<td>➢ Basic rules of differentiation, including the natural exponential rule.</td>
<td>➢ How to differentiate functions involving hyperbolic functions.</td>
</tr>
</tbody>
</table>

I hope you will not be surprised if I tell you that this section is rather short, since the derivatives of hyperbolic functions are few and easy to find.

*I am not surprised, but I am delighted! Let’s see them.*

**Technical fact**

The derivatives of the basic hyperbolic functions are as follows:

\[
\begin{align*}
  (\sinh x)' &= \cosh x \\
  (\cosh x)' &= \sinh x \\
  (\tanh x)' &= \text{sech}^2 x
\end{align*}
\]

**Proof**

Oh, I will not deny you the pleasure of constructing those proofs by yourself. After all, you only need the differentiation rules you have seen so far 😊.

So, off you go to the Learning questions!
Summary

➢ The derivatives of hyperbolic functions can be easily obtained by using their defining formulae and the basic rules of differentiation.

Common errors to avoid

➢ Although the differentiation rules for hyperbolic functions are similar to those of trigonometric functions, they are not exactly the same: do not confuse them!

Learning questions for Section D 5-3

Review questions:

1. Explain why the derivatives of hyperbolic functions are so easy to obtain.

Memory questions:

1. What is the derivative of $y = \sinh x$?
2. What is the derivative of $y = \cosh x$?
3. What is the derivative of $y = \tanh x$?
Computation questions:

Compute the derivative of the functions presented in questions 1-22.

1. \( y = \cosh(\sinh(x)) \)
2. \( y = \frac{\sinh x}{\tanh x} \)
3. \( y = (\cosh^2 x) \sinh x \)
4. \( y = \cosh^2 (\sinh^2 (x)) \)
5. \( f(x) = 3^{\cosh(x)^2+2} \)
6. \( y = \cosh(e^x + \ln x) \)
7. \( y = \frac{\sinh(\ln x)}{\cosh^2(x^2+1)} \)
8. \( y = e^x + \frac{8x}{e^x} - \tanh(3x) \)
9. \( y = e^x \cosh x \)
10. \( y = \tanh \left( \ln \sqrt{\frac{1+x}{1-x}} \right) \)
11. \( f(x) = \log_8 \frac{e^{2x} + x}{x^2 - \sinh x} \).
12. \( f(x) = (2\sqrt{x^2 + 2} - \tanh x) \sin e - \frac{6}{x^3} + 4^x \cos x \)
13. \( y = e^{3x^2} \sinh^3 x + 4 \tanh 2x \)
14. \( y = \frac{\sinh x - \cosh x}{\sinh x + \cosh x} \)
15. \( y = 4x^2 \cosh^3 x + 2 \coth 4x \)
16. \( f(x) = \frac{2\sqrt{x^2 + \ln x} - \tanh x}{8} \)
17. \( f(x) = \log_3 \left( x^2 e^{x^2} \right) + \cosh (x^2 + 4) \)
18. \( f(x) = \frac{2-e^{x^2}}{\cosh \sqrt{x}} \)
19. \( f(x) = \sinh(\cosh 3^x) \)
20. \( f(x) = (\sinh x^2)^x \)
21. \( f(x) = \frac{x^9}{\cosh^8 (1-e^x)} \)
22. \( y = \frac{\cosh^8 (1-e^x)}{x^9} \)

23. Determine the derivative of the function \( y = \frac{\cosh x^2}{e^{x^2}} \) in the following two ways:
   a) By applying appropriate rules to the function as is and simplifying the result algebraically (that is, no need to change the hyperbolic functions).
   b) By rewriting the function in terms of a single exponential and computing the derivative of the resulting form.
24. Find \( dy/dx \) if \( x \cosh y = y + x \)

**Theory questions:**

1. What is the 100\(^{th} \) derivative of \( f(x) = \cosh 2x \)?
2. What method is used to obtain the derivative of the two basic hyperbolic functions?
3. What is the derivative of \( y = \cosh x \sech x \)?

**Proof questions:**

1. Prove that \( (\sinh x)' = \cosh x \)
2. Prove that \( (\cosh x)' = \sinh x \)
3. Prove that \( (\tanh x)' = \sech^2 x \)
4. Compute the derivative of the other three main hyperbolic functions: \( y = \sech x, \ y = \csc x, \ y = \coth x \) starting from their formulae in terms of exponential functions.
5. Compute the derivative of the other three main hyperbolic functions: \( y = \sech x, \ y = \csc x, \ y = \coth x \) starting from their formulae in terms of hyperbolic sine and cosine.

**Application questions:**

For each of the curves presented in question 1-4, determine the equation of the line tangent to it at the given point.

1. \( f(x) = e^{\ln(x^2 + 1) + \cosh x} \) at its \( y \)-intercept
2. \( y = \ln x^2 - \sinh x \) at \((1, -\sinh 1)\).
3. \( f(x) = \frac{\cosh x}{\sinh x - \cosh x} \) at its \( y \)-intercept.
4. \( x \sinh(x) + y \cos y = \ln 4 \) at the point of coordinates \((\ln 2, \ln 2)\).
5. The instantaneous rate of fuel consumption of a car (in the appropriate units) is given by the function \( c(v, a) = 2 + \sinh v + \cosh a \), where \( v \) is the car’s velocity and \( a \) is its acceleration. If the car’s velocity is given by \( v = \frac{t^2}{1+t^2} \).

1) Which function of \( t \) represents the car’s rate of fuel consumption, and is this function differentiable?
2) What is the limiting consumption of the car as \( t \) increases?

**Templated questions:**

1. Construct a function involving hyperbolic functions, determine its derivative, its tangent line at some point and its higher derivatives, as much as reasonably possible.

**What questions do you have for your instructor?**