Chapter 7: Linear approximations — **Section 1**

Linearization of a function

What you need to know already:	What you can learn here:
➤ How to compute derivatives and how they relate to tangent lines and differentials.	➤ How to use tangent lines to approximate complicated functions.

Let me start from something we have seen already in the section on differentials. The definition of derivative at a point tell us that if a function $f(x)$ is differentiable at $x = c$, then:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

By using the meaning of limit, we can interpret this as follows.

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If $y = f(x)$ is differentiable at a value *c* and if *h* is a small number, then

$$f'(c) \approx \frac{f(c+h) - f(c)}{h}$$

This is an approximate equality and it remains so if we apply basic algebra operations. This leads us to another interesting interpretation.

Technical fact

If $y = f(x)$ is differentiable at *x*=*c* and if *h* is a small number, then the formula

$$f(c+h) \approx f'(c)\,h + f(c)$$

allows us to ***estimate*** the value of the function for values of *x* near *c* ($f(c+h)$) by using the simpler formula $f'(c)\,h + f(c)$.

Proof

If we start from $f'(c) \approx \dfrac{f(c+h) - f(c)}{h}$ and multiply both sides by *h*, we get $f'(c)h \approx f(c+h) - f(c)$. By moving one term to the other side we conclude that:

$$f'(c)h + f(c) \approx f(c+h) \quad \Leftrightarrow \quad f(c+h) \approx f'(c)\,h + f(c)$$

Example: $f(x)=\sqrt[3]{x}$; $c=1000$

Since the derivative of this function is $f'(x)=\dfrac{1}{3\sqrt[3]{x^2}}$, by using the above formula we obtain:

$$\sqrt[3]{1000+h}\approx\sqrt[3]{1000}+\frac{1}{3\sqrt[3]{1000^2}}h=10+\frac{h}{300}$$

We can use this formula to estimate manually, for instance, $\sqrt[3]{995}$:

$$\sqrt[3]{995}=\sqrt[3]{1000-5}\approx10+\frac{-5}{300}=10-\frac{1}{60}=9.98\bar{3}$$

If you use your calculator, you will get a value of 9.983305478… , which is very close to what we found with a few easy computations. Not bad, eh?

Wait: is it acceptable to use negative values of h?

Yes: *h* must be small, but it can be positive or negative. Try the same method to estimate the value of $\sqrt[3]{1003}$ and compare that to the value of the calculator.

How small can h be?

That's an excellent question, and like many other excellent questions, does not have a clear answer.

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The formula $f(c+h)\approx f'(c)h+f(c)$ provides a good estimate for small values of *h*.

However, ***how small*** *h* can be ***depends*** on the function (especially how fast it curves near *c*) and what we accept as *good*.

Assessing how good an approximation is, and how to properly define *good*, is an important problem in mathematics: maybe you will encounter it again later.

But for now, let us just take one small step to formalize the concept further. It's a small step for this section, but a giant leap for computational techniques.

Definition

If $y=f(x)$ is a function that is differentiable at $x=c$, then the function:

$$L_f(x)=f'(c)(x-c)+f(c)$$

is called the ***linearization*** of $f(x)$ near *c*.

Technical fact

The linearization $L_f(x)$ provides a ***linear approximation*** of $y=f(x)$ for values of *x* near *c*.

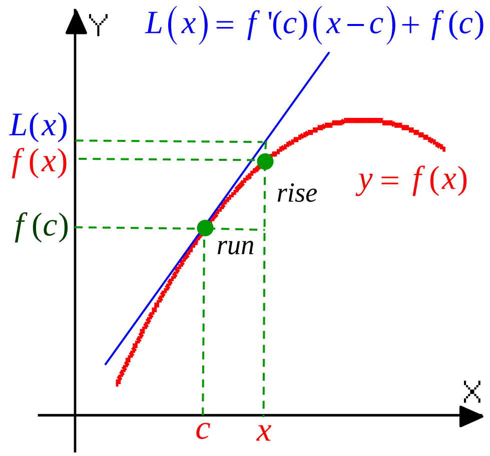
Proof

All we need to do, starting from our linear approximation formula, is to make a substitution. If *x* is near *c*, then $x=c+h$ for some small value of *h*. But that means that:

$$L_f(x)=f'(c)(x-c)+f(c)=f'(c)h+f(c)\approx$$
$$\approx f(c+h)=f(x)$$

This shows that we do indeed have a reasonable approximation.

The geometric situation we are dealing with is illustrated in this picture:



Example: $f(x)=\sqrt[3]{x}$; $c=1000$

We know that $f(1000)=\sqrt[3]{1000}=10$ and, as we have seen before, $f'(1000)=\frac{1}{3\sqrt[3]{1000^2}}=\frac{1}{300}$. Therefore, the linearization of $y=\sqrt[3]{x}$ near 1000 is given by:

$$L_f(x)=\frac{1}{300}(x-1000)+10$$

This function can be used to approximate the cube root function for numbers that are close to 1000: the closer the better.

For instance, here are a few examples, with the calculator value in brackets:

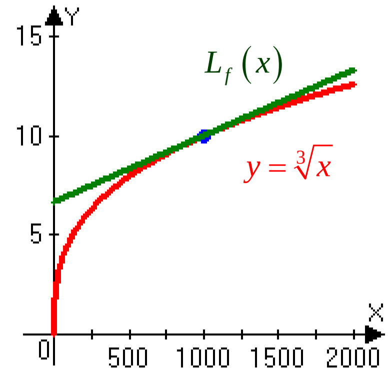
$$L_f(999)=\frac{1}{300}(999-1000)+10=9.99\bar{6} \quad ; \quad [9.9966555...]$$

$$L_f(1010)=\frac{1}{300}(1010-1000)+10=10.0\bar{3} \quad ; \quad [10.033222...]$$

$$L_f(900)=\frac{1}{300}(900-1000)+10=9.\bar{6} \quad ; \quad [9.65489...]$$

$$L_f(1250)=\frac{1}{300}(1250-1000)+10=10.8\bar{3} \quad ; \quad [10.77217...]$$

As you can see, the approximation gets worse as we move further from 1000, but it is still pretty good as far as 1250. A graph visualizes this.



Do not expect such stellar performances always: the cube root function does not curve much near 1000 and that is what makes this approximation so good.

So, can we use this linearization to estimate a function by hand?

Well, that is exactly its reason for being!

Why can't we simply use a calculator?

How do you think a calculator computes functions such as cube roots, trigonometric, exponential and other strange functions?

By using linearizations?

No, but by using more efficient methods developed in an attempt to push the method of linearization further. That is why I said that this was a small step here (an easy proof and formula) but a giant leap for computational methods: it is the motivation that led people to devise the methods now used by our calculators.

Do we call this method linearization and linear approximation because they work best where the function does not curve much?

Yes and no. That is a good justification, but haven't you realized yet what this linearization is all about? Look at the formula again…

It's the tangent line!

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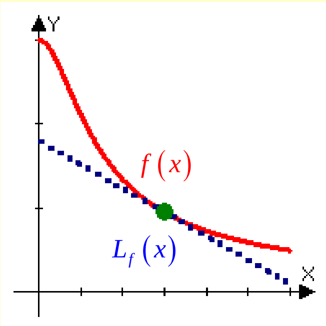
The ***linearization*** $L_f(x)$ of $f(x)$ at $x=c$ is the ***tangent line*** there, but ***with a purpose***: that of being an approximation for $f(x)$, since if x is near c, $L_f(x) \approx f(x)$, as this graph illustrates.



Notice that the equation of the line tangent to $f(x)$ at $x=c$ is $y = f'(c)(x-c)+f(c)$, which is exactly the linearization L_f of $f(x)$ at $x=c$.

That is nice, but this method works only if the computations of $f(c)$ and $f'(c)$ are easy!

Yes, but in real applications, we would start from a difficult value and would have the freedom to use an easy one nearby.

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To use the linearization function ***effectively***, choose the value of c so that the resulting computations are ***simple***.

Example: Estimate $\sin\frac{3}{4}$

To get a numerical and approximate value of this we can use the linearization method, but at which point? We notice that:

$$3 \approx \pi \quad \Rightarrow \quad \frac{3}{4} \approx \frac{\pi}{4}$$

But calculations are easy for trigonometric functions at $\frac{\pi}{4}$, so we can use this as the c value. Now notice that:

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \quad (\sin x)' = \cos x \quad ; \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Therefore the linearization is:

$$L_f(x) = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}$$

To compute $\sin\frac{3}{4}$ we can also approximate the values of $\sqrt{2} \approx 1.414$ and of $\pi = 3.14$. After all, we are *estimating* and therefore it is acceptable to use more approximations, as long as we point them out.

In this way we can compute by hand that :

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = 0.707 \quad ; \quad \frac{\pi}{4} \approx \frac{3.14}{4} = 0.785$$

and conclude that:

$$\sin 0.75 \approx 0.707(0.75 - 0.785) + 0.707 \approx 0.6823$$

The calculator provides a value of 0.68163876…, still pretty close.

And now we can use the same linearization for other values close to $\pi/4$. Try your hand at it: it's a good computation exercise!

Example: Estimate $\ln\sqrt{3}$

Firstly, we need to decide on a function to use, but the number we need clearly suggests the use of $f(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$. Next we need to choose a suitable value for c. Which number do we know that is close to 3 and works well for the natural logarithm? Why, *e*, of course!

So, we evaluate the function and its derivative at this value:

$$f(x) = \frac{1}{2}\ln x \quad \Rightarrow \quad f(e) = \frac{1}{2}$$

$$f(x) = \frac{1}{2}\ln x \quad \Rightarrow \quad f'(x) = \frac{1}{2x} \quad \Rightarrow \quad f'(e) = \frac{1}{2e}$$

We are now ready to construct the needed linearization:

$$L_f(x) = \frac{1}{2e}(x-e) + \frac{1}{2}$$

and we can use it with $x = 3$. On the way to our final numerical estimate we shall need to make some more approximations, which is fine, since we are estimating. By letting $e \approx 2.7$ we have:

$$L_f(3) = \frac{1}{2e}(3-e) + \frac{1}{2} \approx \frac{1}{2\times 2.7}0.3 + 0.5 = \frac{0.3}{5.4} + 0.5 = \frac{0.1}{1.8} + 0.5$$

We now approximate the remaining fraction:

$$\frac{0.1}{1.8} + 0.5 \approx \frac{0.1}{2} + 0.5 = \frac{1}{20} + 0.5 = 0.05 + 0.5 = 0.55$$

How good is this estimate? The calculator's value is 0.54930…, so ours is not bad at all!

Is our calculator's value exact then?

No, it is also an approximation, but you can consider it correct to the decimal digits shown.

And how far can we go with our additional approximations?

The less, the better, but feel free to use reasonable ones. The important thing, however, is to be aware of all the approximations used.

Warning bells

When you use approximation methods, make sure to ***clearly indicate*** the steps where you are using an approximation, by using the symbol $\approx$, and those where you are using an exact equality, by using the symbol $=$.

Summary

- The so-called linearization of a function is simply its tangent line, at a point where it is easy to construct it, with the purpose of estimating values of the function close to it.

Common errors to avoid

- Don't be frightened by the weird name "*linearization*": it simply denotes the tangent line, but with a purpose.
- Always clearly indicate at each step whether you are using an approximation or an exact equality.

Learning questions for Section D 7-1

Review questions:

1. Explain the difference between the linearization of a function at a point and its tangent line at that same point.

2. Describe how to construct the linearization of a function.

3. Describe how to use linearization to estimate the *y* value of a function at a numerically challenging *x* value.

Memory questions:

1. Which formula identifies the linearization of a function $y = f(x)$ at $x = a$?

2. Which special line provides the linearization of a function at a point?

3. What is another name for the function that provides the line tangent to $y = f(x)$ at $x=a$?

Computation questions:

For each of questions 1-8, construct the linearization of the given function at the given point and use it to estimate the required value. Then compare this estimate to your calculator's value. As you solve each question, clearly identify where you are using an equality and where you are using an approximation.

1. $f(x) = \sqrt{45 + x^2}$, $c = 2$, estimate $\sqrt{46}$.

2. $f(x) = \sqrt[3]{x}$; $c = 8$, estimate $\sqrt[3]{9}$.

3. $f(x) = \dfrac{1}{\sqrt{4 - x}}$, $c = 0$, estimate $\sqrt{\dfrac{2}{7}}$.

4. $f(x) = \dfrac{4}{\sqrt{3 + x}}$, $c = 1$, estimate $f(0.99)$.

5. $f(x) = \dfrac{2\cos x}{2 + \sin x}$, $c = \pi$, estimate $f(3)$.

6. $y = \sec x$, $c = \dfrac{\pi}{4}$, estimate $\sec 0.8$.

7. $f(x) = xe^x$, $c = 1$, estimate $1.1e^{1.1}$.

8. $f(x) = x^2 \ln x$, $c = e$, estimate $2.5^2 \ln 2.5$.

In questions 9-12, use the given function and a suitable value of c to estimate the required value. Please notice that here the method is not really efficient, since long division would work better! These questions are designed to help you practice, while keeping the computations simple.

9. $f(x)=\frac{x}{4-x}$, estimate $\frac{1}{39}$.

10. $f(x)=\frac{x}{3-x}$, estimate $\frac{3}{7}$.

11. $f(x)=\frac{x}{\sqrt[3]{x^2-1}}$, estimate $f(\pi)$.

12. $y=\tanh x$, estimate $\tanh 0.7$.

Use a suitable linear approximation to estimate each of the values given in questions 13-18.

13. 2.02^5

14. $\tan 1$.

15. $\ln 3$.

16. $\tanh \ln 3$

17. $\sin 1+\cos 1$.

18. $\sin 1$

19. A function $y=f(x)$ is such that $f'(x)=e^{-x^2}$ and $f(1)=3$. Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$ and decide whether such estimate are higher or lower than the exact values.

20. Construct a suitable linearization of the function $r=p\ln p+\frac{p^2}{2}-1$ and use it to estimate the value of r when p=8.

21. Compute the derivative of the function $f(x)=(\cos x+1)^{\frac{x^3-2x}{\cosh x}}$ and use it to construct a linear approximation suitable to estimate $f(0.1)$. I expect you to compute such estimate by hand, but you will need to obtain ONE approximate value from the calculator: feel free to do so to two decimal digits.

22. Construct a linearization of the parametric curve $\left(-e^t, \frac{1}{e^{3e^t}}+\sin 2e^t\right)$ suitable to estimate the value of y when $x=\frac{\pi}{12}$ and estimate such value.

23. Construct a linearization of the parametric curve $\left(t^3, e^{3t^3}-\sin 6t^3\right)$ suitable to estimate the value of y when $x=\frac{\pi}{12}$ and estimate such value.

Theory questions:

1. In general, which line provides the linear approximation to a function at a point?

2. For what values of x does the linear approximation to a function near a point provide a good estimate of the function?

3. What function represents the linear approximation to $y = 3x + 1$ at (0, 1)?

4. If the second derivative of a function near a certain point is big, will the linearization provide a good approximation or not?

5. Physicists often use the fact that for small values of x the function $y = \sin x$ can be well approximated by the function $y = x$. Why is this approximation valid?

6. Generally speaking, in what situations will a linear approximation work well and in what situations will it give poor estimates?

7. Does a linear approximation work better when the second derivative is small or large?

8. The approximation formula has been given in this section according to the point-slope formula as $L_f(x) = f'(c)(x-c) + f(c)$. However it can also be written in the equivalent form $L_f(x) = f(c) + f'(c)(x-c)$. What geometrical interpretation is highlighted by this form?

<u>Templated questions:</u>

1. Construct the linearization of a function of your choice at a value where it is simple to do so.

2. For any estimate based on a linearization, determine whether what you obtain is an overestimate or underestimate.

What questions do you have for your instructor?