Chapter 8: Graphical analysis | Section 3

Second derivative analysis

What you need to know already:	What you can learn here:
➤ How to perform a first derivative analysis and interpret its result as information about the up/down pattern of the graph.	➤ How to analyze the second derivative and interpret the results as information about the concavity of the graph.

- The first derivative provides information about the increase/decrease pattern of a function.
- The second derivative is the derivative of the first derivative: don't get confused by the words!

These two facts, together, imply the following.

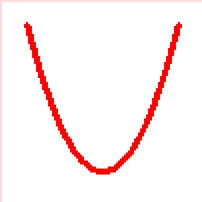
Knot on your finger

The ***second derivative*** provides information on the increase/decrease pattern of the first derivative, that is, of the slope.

But what does that mean for the original function? Knowing how the slope is changing tells us how the graph is *curving*. Here is how.

Technical fact

If $f''(x) > 0$ then the slope of the curve is increasing, so that the curve is ***concave up*** and will present a shape similar to a section of the graph shown here.



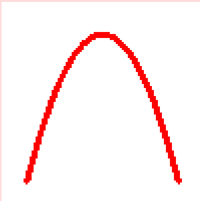
Proof

Not much to prove, since being concave up actually ***means*** having an increasing slope! Just notice that as you follow the graph shown from left to right, the slope does indeed get bigger, whether the function is decreasing (the negative slope becomes less negative) or increasing (the positive slope becomes more so).

And of course the opposite situation is symmetrically true.

Technical fact

If $f''(x) < 0$ then the slope of the curve is decreasing, so that the curve is ***concave down*** and will present a shape similar to a section of the graph shown here.

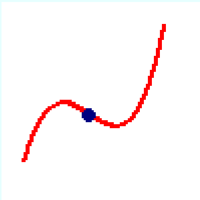


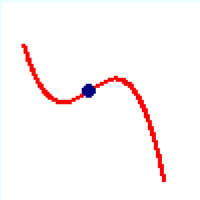
These two facts take care of the points where $f''(x) > 0$ or $f''(x) < 0$, so the only remaining cases relate to where the second derivative is 0 or undefined. At these points the concavity may change and we need one last definition.

Definition

A point $(c, f(c))$ on the graph of a function is an ***inflection point*** if:

- the function is ***continuous*** at $x = c$ and
- the second derivative ***changes sign*** as *x* goes from the left to the right of *c*.





I remember that in high school they told us that an inflection point was one where the second derivative was 0: does it have to be?

That is an interesting point on which you will probably see different opinions. Some authors require that the second derivative be continuous at an inflection point, thus limiting the definition to the case where $f''(x) = 0$. They do that to avoid having to call *inflection point* a point where the curve takes a sharp change. I understand the feeling, which is based on a *technical* issue. But I prefer to focus on the *purpose*: why do we want to identify a point as being an *inflection* point? The most practical answer is that the concavity changes there, so I stay with the more generous definition that allows for the second derivative to be undefined there. It is the point's perspective on the concavity that is of interest in an inflection point.

We are soon going to see some examples, but let me conclude the theoretical part first, by summarizing the strategy for a second derivative analysis. If you get feelings of déjà vu, it's because you have seen the same strategy not long ago!

Strategy for performing a Second derivative analysis

To determine the ***pattern of concavity*** in the graph of $y = f(x)$ and to identify its inflection points :

1. Use standard algebraic methods to ***find the cut points*** of $f''(x)$.
2. ***Place*** these values on a ***number line***, as in the strategy to solve an inequality.
3. ***Test*** each resulting interval, to see if the function is concave up or down there, and indicate this information on the number line.
4. ***Follow the*** concavity ***pattern*** and use any other feature of the function to ***classify*** each cut point as an inflection point or another feature.

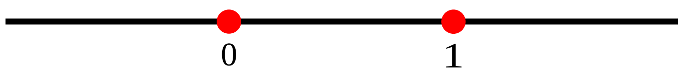
Example: $f(x)=\dfrac{x^3}{x-1}$

To perform a second derivative analysis on this function we begin by computing it. You can check that it is given by :

$$f''(x)=\frac{2x\left(x^2-3x+3\right)}{(x-1)^3}$$

The only x value that makes this fraction undefined is x=1, which we can recognize as corresponding to a vertical asymptote, because of the #/0 form in the original function.

The x values that make it 0 are the roots of the numerator. However, the quadratic factor has no roots and is always positive, so we are left only with $x=0$. Therefore, the cut points of the second derivative are $x=0,1$, which we place on the number line.



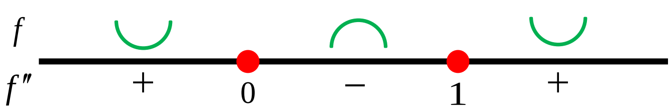
To test the three intervals so obtained, we reason as follows. The quadratic factor and the number 2 are always positive, so we only need to check the remaining two factors, that is, the expression $\dfrac{x}{(x-1)^3}$.

If $x<0$ both factors are negative, making the fraction positive.

If $0<x<1$ the numerator is positive, but the denominator is negative, making the fraction negative.

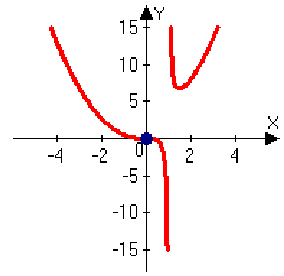
If $x>1$ both factors are positive, making the fraction positive.

Therefore the line graph can be completed as follows:



The function changes concavity at both cut points, however, at $x=1$ we have a V.A. Therefore the only inflection point is at $x=0$, that is, at the origin.

A calculator graph supports the accuracy of this analysis.



Example: $y=\ln\left(x^3+27\right)$

First we notice that this function is defined only if:

$$x^3+27>0 \quad\Rightarrow\quad x>-3$$

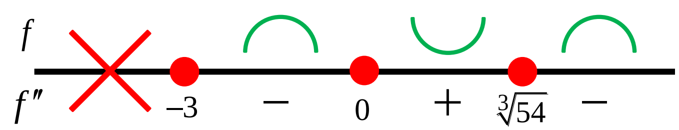
Now we compute the second derivative:

$$y'=\frac{3x^2}{x^3+27} \quad\Rightarrow\quad y''=\frac{6x\left(x^3+27\right)-9x^4}{\left(x^3+27\right)^2}=-\frac{3x\left(x^3-54\right)}{\left(x^3+27\right)^2}$$

This is undefined at the end point $x=-3$ and is 0 when:

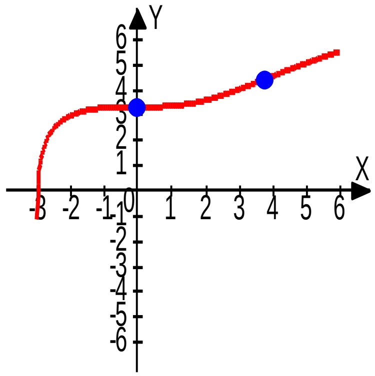
$$x\left(x^3-54\right)=0 \quad\Rightarrow\quad x=0,\sqrt[3]{54}\approx 3.78$$

Considering that the denominator of this derivative is always positive, we obtain the following line graph:



Therefore we have inflection points at 0 and 3.78.

Notice that in this case a calculator's graph clearly shows the first inflection point, but the second one may not be very visible. Following is what a computer graph shows.



The second derivative analysis also allows us to identify a special feature of a graph that may not be identifiable by using only the first derivative analysis.

<u>Definition</u>

A ***cusp*** occurs in the graph of a function at a point where:

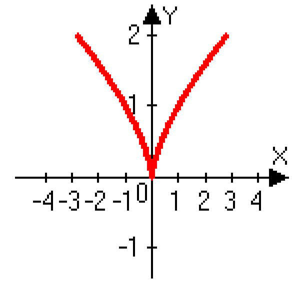
- The function is continuous,
- The tangent line is vertical,
- The concavity does not change.

Example: $f(x)=\sqrt[3]{x^2}$

The first and second derivatives are as follows:

$$f(x)=x^{\frac{2}{3}} \quad\Rightarrow\quad f'(x)=\frac{2}{3}x^{-\frac{1}{3}} \quad\Rightarrow\quad f''(x)=-\frac{2}{9}x^{-\frac{4}{3}}$$

The negative exponents of the two derivatives tell us that both become infinite at $x=0$. However, the function is continuous there and the derivative is positive on both sides of it. The graph clearly shows the "cuspy" nature of the point.



Are inflection points and cusps the only features that can occur at a critical value for the second derivative?

The straight answer to your question is NO, but be careful! In wording your question you included a common error and seemed to ignore something we already saw in the first derivative analysis. First of all…

<u>Warning bells</u>

A ***critical value*** is a feature of a function related to its ***first derivative*** only. The cut points of the second derivative do not have a special name.

Oops, sorry! And what did I forget?

<u>Knot on your finger</u>

A cut point of the second derivative may correspond to a ***discontinuity***. In particular it may correspond to a vertical asymptote.

Oh, right! Is that it then?

As I said, the answer to your questions is no, as there are other possible features. One in particular is worth noting.

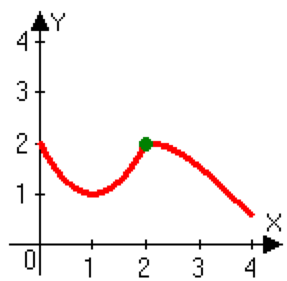
Technical fact

A cut point of the second derivative may also correspond to a point where the function is continuous, but has a tangent line coming from the left side that is different from the one obtained coming from the right side.

Such points are sometimes referred to as ***sharp points*** and you will see them here only in connection to piecewise functions, although they may occur in other, more complex functions as well.

Example: $f(x)=\begin{cases} x^2-2x+2 & \text{if } x\le 2 \\ \cos(x-2)+1 & \text{if } x\ge 2 \end{cases}$

The graph of this function is shown here.

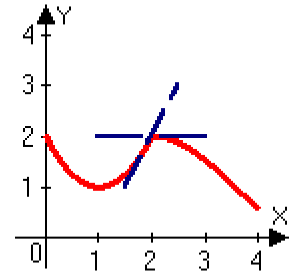


You can see that at the point $(2,2)$ there is a sharp change of direction. That is why we call this a sharp point. Notice that in this case $(2,2)$ is also an inflection point, but this may not be the case in other situations.

Also, by computing the first and second derivative, you can check (do it!) that the function has different slopes if we look at the left and right side, that is:

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}\neq \lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$

Hence we can think of there being two different tangent lines at this point, as shown here. Of course the function has no unique tangent line there, as the tangent line must be unique to exist, but it is possible to refer to a left and right tangent.



I will conclude this section by mentioning a popular, but rather ineffective use of the second derivative.

Technical fact:

The Second Derivative Test

If $x=c$ is a critical value of $y=f(x)$, then:

- If $f''(c)>0$, then $(c, f(c))$ is a ***relative minimum*** for $f(x)$.
- If $f''(c)<0$, then $(c, f(c))$ is a ***relative maximum*** for $f(x)$.

The proof of this fact is so simple that I am giving it to you as a *Proof question*.

You said that it is popular, but ineffective: why?

It is popular because it appeals to mathematicians: it is a test that manages to link two seemingly disparate concepts like the second derivative and critical values, which relate to the first derivative.

However, it is an inefficient use of the second derivative, because the test requires us to compute and use the second derivative when in fact the first derivative is sufficient to obtain its conclusions. Moreover, this test tells us nothing about what happens at other critical values where the second derivative is 0 or undefined, while the first derivative analysis approach that we have used gives us an answer in all cases.

Therefore, while I expect you to know what the *Second Derivative Test* is, I will not expect you to use it in the classification of critical values.

Summary

- The second derivative analysis provides information about the concavity of the graph and its inflection points.
- It uses the same method as the first derivative analysis, based on the method to solve inequalities.

Common errors to avoid

- Do not confuse the second derivative analysis with the second derivative *test*, whose purpose is to classify critical values.

Learning questions for Section D 8-3

Review questions:

1. Describe the purpose of a second derivative analysis.
2. Explain how a second derivative is implemented.
3. Identify the difference between a second derivative analysis and a second derivative test.

Memory questions:

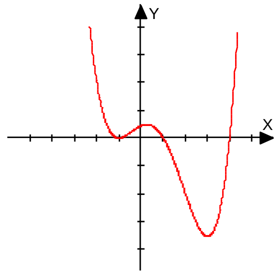
1. What does a positive second derivative imply for the graph of a function?
2. What does a negative second derivative imply for the graph of a function?
3. What identifies an inflection point?
4. Which cut points for the second derivative are inflection points for the graph?
5. What is the purpose of the *Second Derivative Test*?

Computation questions:

Perform a full second derivative analysis on the functions presented in questions 1-19.

1. $y = x^5 - 10x^3$
2. $y = x^4 - 4x^2$
3. $y = \dfrac{x+1}{x^2 - x}$
4. $y = \dfrac{1}{x^3 - x}$.
5. $f(x) = \dfrac{x^3 + 2}{x^3 - 8}$
6. $f(x) = \dfrac{x+1}{x^2 - 4}$.
7. $f(x) = (x-1)^{2/3} - (x+1)^{2/3}$
8. $f(x) = x^{-\frac{2}{3}} + 2x^{\frac{1}{3}}$
9. $f(x) = \dfrac{2x}{\sqrt{x^2 - 5}}$
10. $f(x) = \dfrac{x-1}{\sqrt{x^2 + 2}}$
11. $f(x) = \dfrac{1+2x}{\sqrt[3]{x}}$
12. $f(x) = \dfrac{x^2 - 1}{\sqrt{x}}$
13. $f(x) = \sqrt[3]{x}\left(x^3 - 1\right)$
14. $f(x) = \sqrt{x^2 - x^3}$
15. $f(x) = \dfrac{e^{x^2}}{x}$
16. $y = e^{x^2 - x^3}$
17. $y = \ln\left(x^2 + 2x + 2\right)$
18. $y = x \ln x^2$
19. $y = \sin^2 x + 2\sin x, \ \ 0 \le x \le 2\pi$

20. A function has a derivative whose graph is shown below. From it conduct a first and second derivative analysis of the original function and briefly describe your method. Consider each tick mark as one unit.



Theory questions:

1. Can an inflection point occur at a sharp corner (cusp)?

2. Can an inflection point occur where the first derivative is undefined?

3. Mention one function that has infinitely many inflection points.

4. If at a certain point *y''* is undefined, but it is positive on both side of the point, what two graphical feature can occur?

5. Can the second derivative on either side of a sharp corner be of the same sign?

6. Is every point for which *y*"=0 an inflection point?

7. Can the second derivative equal 0 at a point with a vertical tangent line?

8. Does the Second Derivative Test tell us that the second derivative at a minimum is positive?

9. Can a function have a local maximum at an inflection point?

10. What graphical feature occurs at a point where the function is continuous and its second derivative is undefined, but keeps the same sign?

11. Which algebraic method is the basis for both the first and second derivative analysis?

12. What graphical feature occurs at a point where the a function is continuous and its second derivative changes from positive to negative?

13. What two graphical features may occur at a critical value that does not generate a maximum or a minimum?

14. Which cut points of the second derivative are critical values?

15. Is it possible for a function to have a maximum at a point where the second derivative does not exist?

16. How good is the linear approximation of a twice differentiable function at an inflection point?

17. If the derivative of a function is decreasing, how is the concavity of the function itself?

18. If a function is continuous, but is not differentiable at x=0, does t mean that it has a sharp corner there?

19. If the line tangent t a function at a point seems to cross the curve near that point what feature is occurring?

20. If a function is increasing towards its right horizontal asymptote, and has finitely many inflection points, how will the second derivative be for large values of x?

Proof questions:

1. Prove that the function $y = \dfrac{1}{x^2 + cx}$ has no inflection points, no matter what the value of c is.

2. Prove the *Second Derivative Test*.

Templated questions:

1. Perform a second derivative analysis on each of the functions listed in the document Sample functions to analyze.

2. Perform a second derivative analysis on a reasonably simple function of your choice.

What questions do you have for your instructor?