Chapter 8: Graphical analysis **Section 4**

L'Hospital's rule

What you need to know already:	What you can learn here:
➤ All the basic methods for computing limits and derivatives.	➤ How to use derivatives to resolve certain indeterminate forms.

By using first and second derivative analyses we can find most of the important pattern features of the graph of a function. To obtain a good sketch of such graph all we need to do is add the location information that can be obtained from the function itself, such as intercepts, asymptotes and points of discontinuity.

So, can we now use calculus methods to construct a good sketch of the graph of a function?

Unfortunately, we still have one problem.

Many transcendental functions have indeterminate limit forms that cannot be resolved by using any of the algebraic methods we saw earlier, such as factor-and-cancel or rationalizing, or special properties of such functions. But mathematicians do not give up that easily, so a method was found to compute interesting limits in such situations and this is done by using, of all things, derivatives!

Why the expression of surprise?

Remember that we used limits to define derivatives, and now the favour is returned: we shall use derivatives to compute limits.

Technical fact: L'Hospital's rule

Assume that for two functions $f(x)$ and $g(x)$ and a value *c*, finite or infinite, the following are true:

- $f(x)$ and $g(x)$ are ***differentiable*** in an interval around *c* except possibly at *c*
- $g(x) \neq 0$ in an interval around *c* except possibly at *c*
- $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{0}$ *or* $\mp\frac{\infty}{\infty}$
- $\lim_{x\to c}\frac{f'(x)}{g'(x)} = k$ *or* $\mp\infty$

Then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$.

Proof

The proof of this fact is another exercise in technicalities, but if you are interested in seeing it (at least once in your life!) you can check it at the URL linked here.

Of course we can use this fact to develop a new strategy to resolve indeterminate forms that are resistant to basic methods.

Strategy for L'Hospital's rule

To try to compute a limit of the type:

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{0}{0} \quad or \quad =\frac{\infty}{\infty}$$

1. **Check** that $g(x)\neq 0$ in an interval around c, except possibly at c.
2. **Compute** $f'(x)$ and $g'(x)$.
3. Compute the same limit for the ratio of the two derivatives:
$$\lim_{x\to c}\frac{f'(x)}{g'(x)}=L$$
4. If L is a finite value or infinite, ***then we know*** that:
$$\lim_{x\to c}\frac{f(x)}{g(x)}=L$$

Example: $\lim_{x\to 0}\frac{\sin x}{x}$

We have seen and used this limit often, but we have never seen a formal proof! Well, now we can prove it formally, as long as we accept the validity of L'Hospital's rule!

- This limit is of the form $0/0$.
- The denominator is only 0 at our value.
- The limit of the ratio of the derivatives is:
$$\lim_{x\to 0}\frac{\cos x}{1}=1$$
- Therefore our limit is 1.

Example: $\lim_{x\to 0}\frac{e^x-1}{x}$

We have also seen this limit before, as the limit that defines the derivative of the natural exponential function at $x=0$. We used the definition of derivative and of the number e to conclude that this limit is also 1.

We can now use L'Hospital's rule to confirm that the value is correct. All assumptions are satisfied and:

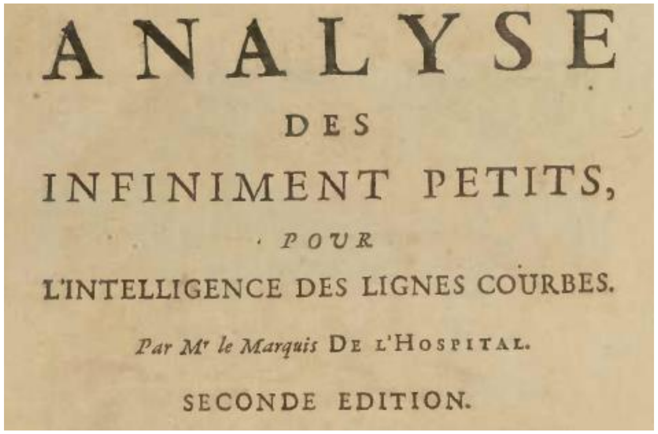
$$\lim_{x\to 0}\frac{(e^x-1)'}{x'}=\lim_{x\to 0}\frac{e^x}{1}=1$$

Yep!

That looks like a fast and efficient method, but what does it have to do with hospitals?

That is a cool story, but about people, not hospitals. The method was first developed by a mathematician named Johann Bernoulli and was used extensively by professionals, before being included in the first calculus textbook aimed at the general public. This book was written by the French Marquis de L'Hospital and, in honour of his work to make calculus available to all, it was named after him.

Incidentally, you may see his name also spelled as L'Hôpital, which today looks more French. However, dropping the "s" has been a relatively recent change in the French language and, as you can see from the title page of his book, the Marquis himself used the old spelling. If you click on the picture, you can see the whole book!



ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par Mr le Marquis DE L'HOSPITAL.
SECONDE EDITION.

And does the method always work?

No! In order for it to work, the limit of the ratio of the derivatives must be a finite value or a confirmed infinity. If it too is of the $0/0$ or ∞/∞ form, we can try again: if it works after a few attempts, we can retrace our way back to the original limit. But we may end up in a repeating cycle of indeterminate forms from which we cannot come out. In that case we need even more advanced methods.

Example: $\lim_{x\to 0}\frac{e^x-1-x}{x^3}$

If we try to evaluate this form, we get a 0/0 form, and the assumptions about *c* are satisfied, so we can apply the rule and start by computing derivatives on top and bottom:

$$\lim_{x\to 0}\frac{\left(e^x-1-x\right)'}{\left(x^3\right)'}=\lim_{x\to 0}\frac{e^x-1}{3x^2}=\left(\frac{0}{0}\right)$$

This is still a 0/0 form that satisfies all assumptions, so we can apply L'Hospital's rule again:

$$\lim_{x\to 0}\frac{\left(e^x-1\right)'}{\left(3x^2\right)'}=\lim_{x\to 0}\frac{e^x}{6x}=\left(\frac{1}{0}\right)$$

This form is NOT indeterminate: in fact we know that the limit is infinite, but we need to check the two sides separately:

$$\lim_{x\to 0^-}\frac{e^x}{6x}=\frac{+}{-}=-\infty;\qquad \lim_{x\to 0^+}\frac{e^x}{6x}=\frac{+}{+}=\infty$$

Since this is true for this last limit, it is also true of the previous one. And if it true for the previous one, it is true for the original limit.

Therefore:

$$\lim_{x\to 0}\frac{e^x-1-x}{x^3}=DNE,\ \ but:$$

$$\lim_{x\to 0^-}\frac{e^x-1-x}{x^3}=-\infty\quad;\quad \lim_{x\to 0^+}\frac{e^x-1-x}{x^3}=\infty$$

Warning bells

Notice that the conclusion of L'Hospital's rule can only be claimed ***after*** $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ has been ***resolved***.

To claim that $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$ before the latter has been resolved is a ***technical foul*** that will be punished harshly! ☺

So, I cannot use that equality while I am still working on the limit?

Correct! It would be like claiming that a goal has been scored in soccer, or hockey, when a penalty shot is called. However, the following convention is often and conveniently used.

Knot on your finger

The notation:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} \overset{H}{=} \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

is used to mean that the equality is true, ***provided*** the second limit can be resolved.

With this notation we indicate that the equality is true only on the condition that L'Hospital's rule works.

By the way, this convention explains why we call this method a *rule*: If we keep the assumptions in the background and use this notation, the whole method reduces to this one step of changing from the original functions to their derivatives. But remember that we are being sloppy when we do this and sloppiness works only if we are very much in control of what we are doing. So, I recommend that you always consider L'Hospital's *rule* as a *method*

I will try to keep that in mind.

But I notice that this method can only be applied to functions that are fractions. That is pretty limiting.

It can be, but there are some situations where we can use our knowledge of algebra (on which you are still working, aren't you?) to make some useful changes.

Technical fact

An indeterminate form of the type:

$$\lim_{x\to c} f(x)\times g(x) = 0\times\infty$$

can be changed to a form suitable for L'Hospital's rule by changing it to one of these forms:

$$\lim_{x\to c}\frac{f(x)}{\left[g(x)\right]^{-1}} = \frac{0}{0} \quad or \quad \lim_{x\to c}\frac{g(x)}{\left[f(x)\right]^{-1}} = \frac{\infty}{\infty}$$

Which alternative form to use depends on the situation and should be checked on a case by case basis.

Example: $\lim_{x\to 0^+}\sqrt{x}\ln x$

This generates a $0\times\infty$ form, so we can change its appearance by moving the root to the denominator. Moving the logarithm will create more difficult functions to differentiate. In this way we get:

$$\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}} = \left(\frac{-\infty}{\infty}\right)$$

This form is now suitable for our method, so we compute the needed derivatives:

$$\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -2x^{1/2} = 0$$

Therefore, our original limit is 0 and the function has … what kind of graphical feature?

Technical fact

An indeterminate form of the type:

$$\lim_{x\to c} f(x) = \infty - \infty$$

may sometimes be changed to a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form by ***combining*** two fractions or two logarithm into a single one with common denominators, or by ***rationalizing*** a difference of two roots.

Example: $\lim_{x\to 1^+}\left(\frac{3}{\ln x} - \frac{2}{x-1}\right)$

This is an indeterminate form of the type $\infty - \infty$ obtained from the difference of two fractions. So we use a common denominator to change the appearance of the function to:

$$\lim_{x\to 1^+}\left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to 1^+}\frac{3x-3-2\ln x}{(x-1)\ln x} = \left(\frac{0}{0}\right)$$

This is now in the correct form and satisfies all assumptions, so we can switch to the ratio of derivatives:

$$\lim_{x\to 1^+}\frac{3-\frac{2}{x}}{\frac{(x-1)}{x}+\ln x} = \lim_{x\to 1^+}\frac{3x-2}{x-1+x\ln x} = \left(\frac{1}{0}\right) = +\infty$$

Conclusion: the original limit is infinite.

Technical fact

An indeterminate form of the types:

$$\lim_{x\to c} f(x)^{g(x)} = 0^0, \infty^0, 1^\infty$$

may be made suitable for L'Hospital's rule by writing the function as:

$$f(x)^{g(x)} = e^{g(x)\ln f(x)}$$

and applying L'Hospital's rule to the exponent only:

$$\lim_{x\to c} f(x)^{g(x)} = e^{\lim_{x\to c} g(x)\ln f(x)}$$

Proof

This is true because the exponential function is continuous and, therefore, the anti-Murphy's law can be used to move the limit to the exponent.

Notice that when we use this fact, we may still need to rewrite the exponent in quotient form before applying the rule.

Example: $\lim_{x\to 0^+}(\sin x)^{1/\ln x}$

This limit takes the form $0^{1/-\infty} = 0^0$, so we change the function to:

$$(\sin x)^{1/\ln x} = e^{\frac{\ln \sin x}{\ln x}}$$

We can now apply L'Hospital's rule to the exponent:

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\ln x} \overset{H}{=} \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\cos x}{\frac{\sin x}{x}} = \frac{1}{1} = 1$$

Therefore, our limit is:

$$\lim_{x\to 0^+}(\sin x)^{1/\ln x} = \mathrm{e}^{\lim_{x\to 0^+}\frac{\ln\sin x}{\ln x}} = e^1 = e$$

See what your grapher shows you in this case.

Instead of moving the limit to the exponent, we can apply an alternative method that uses the same approach of logarithmic differentiation.

Example: $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x, -\infty < a < \infty$

This is an indeterminate form of the type 1^∞ related to the function:

$$y = \left(1+\frac{a}{x}\right)^x$$

Before proceeding, we apply the natural logarithm to both sides:

$$\ln y = \ln\left(1+\frac{a}{x}\right)^x = x\ln\left(1+\frac{a}{x}\right) = \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$

Now we compute the same limit for the function on the right, a task that can be accomplished with L'Hospital's rule, since we have a 0/0 form there:

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{x}{x+a}\left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{ax}{x+a} = a$$

Therefore, we know that $\lim_{x\to\infty}\ln y = a$, and by applying the exponential function and using the Anti-Murphy's law, we conclude that:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a, -\infty < a < \infty$$

By the way: no person who has passed a first calculus course should forget this last limit!

One last word of warning, as it is related to an error frequently made by beginners.

Warning bells

L'Hospital's rule relies on the quotient of the derivative, NOT the derivative of a quotient, so ***do not use the quotient rule*** for the original function!

Summary

- L'Hospital's rule is a method that uses derivatives to compute certain indeterminate limit forms.
- Although it can be applied immediately only to indeterminate forms of the type $0/0$ or ∞/∞ , many other forms may be changed to one of these by using suitable algebraic operations, sometimes involving logarithms.

Common errors to avoid

- When using L'Hospital's rule, we use the quotient of two derivatives, NOT the derivative of a quotient.

Learning questions for Section D 8-4

Review questions:

1. Describe when and how L'Hospital's rule can be used.

Memory questions:

1. To which two indeterminate forms can L'Hospital's rule be applied?
2. How can one apply L'Hospital's rule to an indeterminate form of the type $0 \bullet \infty$?
3. How can one apply L'Hospital's rule to an indeterminate form of the type 0^0, 1^∞ or ∞^0 ?
4. What is the value of $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$?

Computation questions:

Compute the limits of questions 1-8 by applying first basic methods and then L'Hospital's rule. Ensure that the conclusion is the same.

1. $\lim_{x\to -2}\frac{x^3+8}{x+2}$

2. $\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}$

3. $\lim_{x\to 3}\frac{x^2-9}{2x^2-5x-3}$

4. $\lim_{x\to 2}\frac{x^2-4}{x^2+7x-18}$

5. $\lim_{x\to 3^+}\frac{\sqrt{x^2-8}-1}{\sqrt{x+1}-2}$

6. $\lim_{x\to 4}\frac{\sqrt{x^2+9}-5}{\sqrt{x-3}-1}$

7. $\lim_{x\to 0}\frac{\cos \pi x-1}{x}$

8. $\lim_{x\to 0}\frac{\sin 3x}{\tan x}$

Compute the limits of questions 9-28 by properly applying L'Hospital's rule in combination with any other method required.

9. $\lim_{x\to 0}\frac{2xe^x}{\ln(x+1)^2}$

10. $\lim_{x\to\infty}\frac{\ln x}{\cosh x}$

11. $\lim_{x\to\infty}\frac{6x+\cosh x}{x+\sinh x}$

12. $\lim_{x\to -\infty}\frac{6x+\sinh x}{x-\cosh x}$

13. $\lim_{x\to 0}\frac{\tanh\left(e^x\right)-\tanh\left(e^{-x}\right)}{\sinh x}$

14. $\lim_{x\to 0}\frac{\cosh\left(e^x\right)-\cosh\left(e^{-x}\right)}{\tanh x}$

15. $\lim_{x\to 0} x^2\cot x$

16. $\lim_{x\to\frac{\pi}{2}} x\sec x\left(\sin^2 x-1\right)$

17. $\lim_{x\to\frac{\pi}{4}^+}\left(1-\tan x\right)\sec 2x$

18. $\lim_{x\to 0^-}\left(\csc x-\frac{1}{x}\right)$

19. $\lim_{x\to 1^-}\frac{x-1}{\cos^{-1} x}$

20. $\lim_{x\to 0}\frac{x^2}{\tan^{-1} x}$

21. $\lim_{x\to 1}\frac{\sin \pi x+\sin\left(x-1\right)}{e^{x-1}-\cos \pi x-2}$

22. $\lim_{x\to 1}\frac{1+\cos \pi x}{\ln x}$

23. $\lim_{x\to 0}\frac{\sin 2x}{\sinh x}$

24. $\lim_{x\to 0}\frac{\sinh 2x}{\tan x}$

25. $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\ln x}}$

26. $\lim_{x\to 0^+}(\tan x)^{2x}$

27. $\lim_{x\to\infty}\left(1+\frac{3}{x^2}\right)^{2x^2}$

28. $\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^{3x^2}$

Identify and classify all discontinuities of the functions provided in questions 29-41.

29. $y=\frac{3}{e^x-1}-\frac{2}{x^2}$

30. $y=\frac{1}{x}-\frac{1}{xe^{2x}}$

31. $y=(\tan x)^{\frac{1}{4x-\pi}}$

32. $y=\frac{\sin(x-3)}{\ln(x^2-8)}$

33. $y=\frac{1-e^{\tan x}}{\tan x}$

34. $y=\frac{x}{\tanh^{-1}x}$

35. $y=\frac{\cosh x-1}{x-2\sinh x}$

36. $y=\frac{x}{\sinh^{-1}x}$

37. $y=\frac{3x}{\sin^{-1}2x}$

38. $y=\frac{3\sqrt{x}}{\sin^{-1}x}$

39. $y=\frac{\cos 3x-\cos 6x}{9x^2}$

40. $y=\frac{\ln(1-x^2)}{\tan\left(\frac{\pi x}{2}\right)}$

41. $y=\frac{x\ln x}{1+x^2}$

Determine the horizontal asymptotes, in each direction, of the functions presented in questions 42-45, or explain why they do not exist.

42. $y=\left(1-\frac{2}{\sqrt{x}}\right)^{\sqrt{x}}$

43. $y=(e^{-x}-x)^{1/x}$

44. $y=(x-\sqrt{x})^{1/\sqrt{x}}$

45. $y=\frac{x\ln x}{1+x^2}$

Theory questions:

1. Who actually developed L'Hospital's rule?

2. What kind of asymptotes can be identified by using L'Hospital's rule?

3. Mention one condition – different from the type of indeterminate form – that, if present, makes the use of L'Hospital's rule inappropriate.

4. Explain why L'Hospital's rule is not needed to determine the limits of $y=\dfrac{2xe^x}{\ln(x+1)^2}$ as x approaches -2 and -1.

5. If L'Hospital's rule can be applied to $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ and $\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$ does not exist, when does it follow that the original limit does not exist either?

6. In general, what can we say about $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ when $\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$ does not have a limit, either finite or infinite?

7. What technical condition is required by L'Hospital's rule for the function in the denominator?

8. Which fundamental limit that we have used repeatedly, but never proved, can be easily proved by using L'Hospital's rule?

Proof questions:

1. Show that for any natural numbers n and m, $\lim\limits_{x\to\infty}\dfrac{(\ln x)^n}{x^m}=0$

2. Prove that only one of these two limits exists. Evaluate it and explain why the other does not exist.

$$\lim_{x\to\infty}\left(\frac{2x^2-5}{2x^2+4}\right)^{\frac{x}{2}} \qquad \lim_{x\to\infty}\left(\frac{5-2x^2}{2x^2+4}\right)^{\frac{x}{2}}$$

3. Explain why L'Hospital's rule does not help in computing $\lim\limits_{x\to 0^+}\dfrac{e^{-\frac{1}{x}}}{x}$ and find a better way to compute this limit.

Application questions:

1. Show that the parametric curve $\left(\frac{t^2}{e^t}, \frac{2\ln t}{t}\right)$ approaches the origin as t approaches ∞.

2. Determine whether the parametric curve $\left(\frac{t^2}{e^t}, \frac{2\ln t}{t}\right)$ has a vertical asymptote

3. Which point is approached by the parametric curve $\left(\frac{\ln^2 t}{3t}, \frac{t^2}{\cosh t}\right)$ as the parameter approaches positive infinity?

4. Which point is approached by the parametric curve $\left(\frac{\ln^2 t}{3t}, \frac{t^2}{\cosh t}\right)$ as the parameter approaches positive infinity?

5. Compute $\lim_{t\to\infty}\left(\frac{\cosh t}{\ln 2t}, \frac{\cosh t}{e^{2t}}\right)$ and use it to determine which graphical feature occurs in this curve as t approaches ∞.

6. For what values of a and b is the following parametric function continuous?

$$(x, y) = \begin{cases} \left(e^t, (a+b)\cosh t\right) & if\ t \le 0 \\ \left(\frac{\sin at}{bt}, \frac{\sinh t}{t}\right) & t > 0 \end{cases}$$

What questions do you have for your instructor?