Chapter 8: Graphical analysis | **Section 5**

Graph sketching

What you need to know already:	***What you can learn here:***
➤ How to compute and interpret limits ➤ How to perform first and second derivative analyses.	➤ How to gather and use information about a function in order to sketch an informative graph.

Let me get one thing straight right off the bat.

Knot on your finger

The exact graph of a function can only be obtained by ***plotting all*** its points.

Since plotting can only be done on finitely many points, while a graph has infinitely many points, ***every graph*** of a function ***is approximate***, even the ones made by the best computers.

What changes is only the degree of ***accuracy*** of the graph.

Why are you saying this? I am confused!

To help you reach a balance in your search for a graph. The reflection I just offered leads to the following general strategy.

Strategy for a reasonable approach to graph sketching

Since sketching the graph of a function can only be done approximately:

- ***Do not*** become obsessed with wanting to get all details and each ***absolutely correctly***, ***BUT***
- ***Ensure*** that all ***important features*** of the graph are identified, classified and ***clearly visible*** in the graph you construct.

And what should we consider as "important" features?

I am glad you asked! There are three types of important features, and by now you have seen how to find them all. So, here are the three types, one at a time.

Strategy for finding the location features of a function

By focusing on the ***original function*** $f(x)$ and by using algebra and limits methods, we can obtain information on:

- ***Domain***
- ***Intercepts***
- ***Discontinuities***
- ***Behaviour at infinity***

These features help us identify the location of certain key details of the graph.

Strategy for finding the up/down pattern of the function

By focusing on the ***derivative*** $f'(x)$ and by using algebra, limits and differentiation methods, we can obtain information on:

- ***Intervals of increase and decrease***
- ***Extreme points***

And you have probably guessed what the third set of features includes.

Strategy for finding the concavity pattern of the function

By focusing on the ***second derivative*** $f''(x)$ and by using algebra, limits and differentiation methods, we can obtain information on:

- ***Concavity***
- ***Inflection points***

By using this information we can sketch a reasonably informative graph in most cases, and certainly in all cases that you will find in tests.

But there is one more important strategy to consider and this deals with how to put together all the information you find! Many students first collect all the information and then try to visualize the whole graph in one fell swoop. Not a good idea! Instead, try the following.

Strategy for how to build the graph of a function from the information available

To construct an informative sketch of the graph of a function $y = f(x)$, jot down ***each piece*** of information available on the Cartesian plane ***as soon as you find it***. Only at the end ***connect*** all the pieces in a ***reasonable*** and ***consistent*** way.

That's too vague!

That is because the specific details can vary tremendously from function to function. Here are some examples for your perusal.

Example: $f(x) = 4x^2 - x^4$

We start with the function itself. The *y* intercept is:

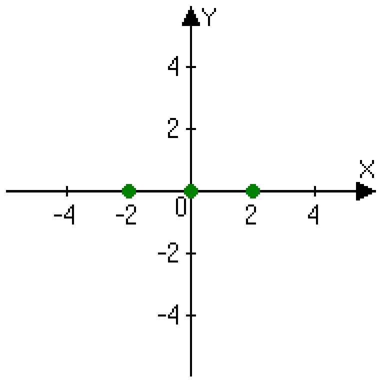
$$x = 0 \quad \Rightarrow \quad f(0) = 0 \quad \Rightarrow \quad (0, 0)$$

The *x* intercepts are:

$$y = 0 \quad \Rightarrow \quad 4x^2 - x^4 = 0 \quad \Rightarrow \quad x^2\left(4 - x^2\right) = 0$$

$$\Rightarrow \quad x = 0, \pm 2 \quad \Rightarrow \quad (0, 0), (-2, 0), (2, 0)$$

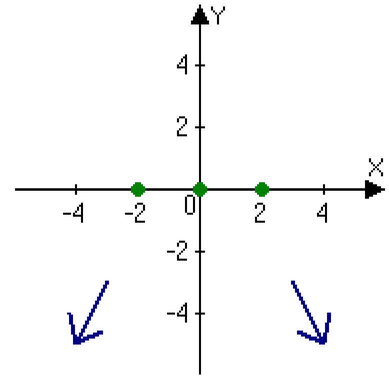
We start placing these points on the graph:



Being a polynomial, this function has no discontinuities, nor horizontal asymptotes. However, we look at the limits at infinity to figure out how the graph exits the visible window.

$$\lim_{x \to \infty}\left(4x^2 - x^4\right) = \lim_{x \to -\infty}\left(4x^2 - x^4\right) = -\infty$$

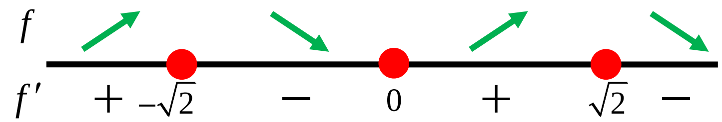
We now add this information to the graph, even though it is rather sketchy now.



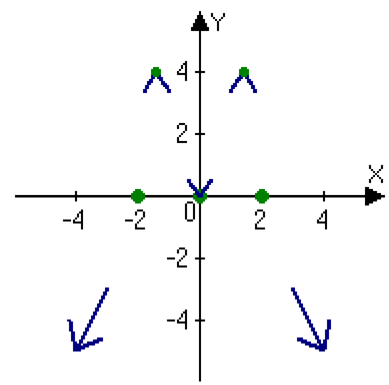
The first derivatives is:

$$f'(x) = 8x - 4x^3 = 4x\left(2 - x^2\right)$$

It produces the following line graph:



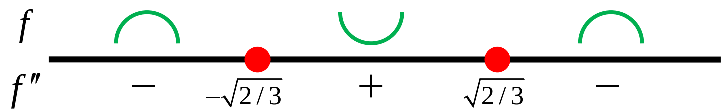
Thus we have relative maxima at $\left(-\sqrt{2}, 4\right)$ and at $\left(\sqrt{2}, 4\right)$. Since they have the same y coordinate and the function does not go any higher, they are also absolute maxima. At the origin we have a minimum. We add this information to the graph, in the form of small wedges around the points, since we do not have information about the second derivative yet.



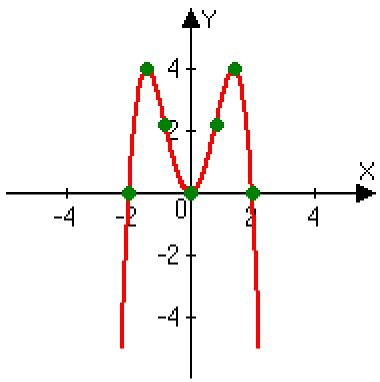
The second derivative is:

$$f''(x) = 8 - 12x^2 = 4\left(2 - 3x^2\right)$$

It produces the following line graph:

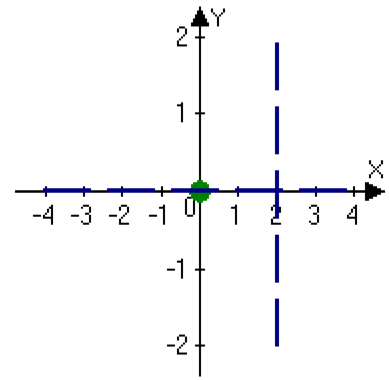


Therefore we have two inflection points at $x \approx \pm 0.82$. We indicate these as well on the graph and complete it by joining all the points and using all the features.

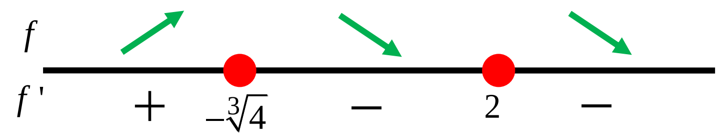


Example: $y = \dfrac{x}{x^3 - 8}$

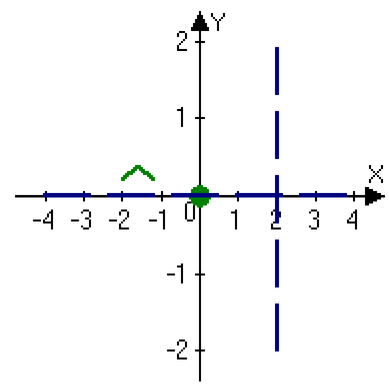
We can easily see that the only intercept of this function occurs at the origin. Moreover, the law of the jungle tells us that there is a HA at $y = 0$ and the denominator tells us that $x = 2$ is a vertical asymptote. We begin to place all these items on the graph. Actually, by sneaking a peak at the calculator's graph, we can choose a suitable window for our graph:



The first derivative is $y' = \dfrac{-2x^3 - 8}{\left(x^3 - 8\right)^2}$ (check it!) and hence its line graph is:



Therefore, the critical value at $x = -\sqrt[3]{4}$ is a relative maximum. We add this information to the graph.

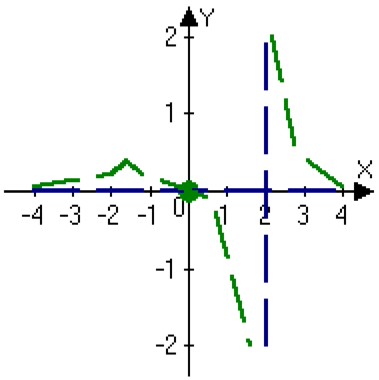


In fact, we can add more information:

- As $x \to -\infty$ the graph has no more *x*-intersections, so it must approach the horizontal asymptote from above.

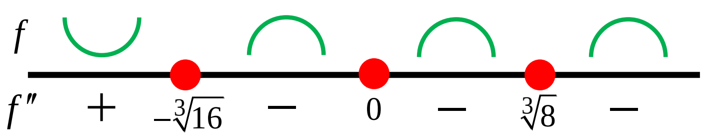
- The function is decreasing both to the right and to the left of the vertical asymptote.
- As $x \to \infty$ the graph has no more x-intersections, so it must approach the horizontal asymptote from above.

We can add this information to the graph as well:



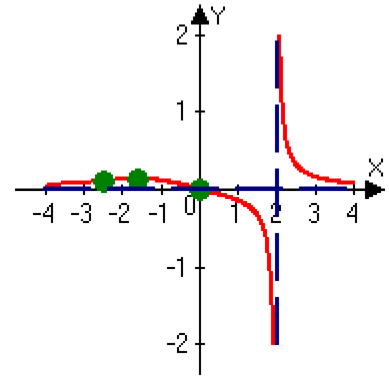
This is still very rough and we can improve on it by adding second derivative information. Notice also that the scale is not right: the maximum is higher than where it should be, but if we place it in the proper position it is hardly visible! That is acceptable, since we are producing a *sketch* only.

The second derivative is $y'' = \dfrac{-6x^2\left[-x^3-16\right]}{\left(x^3-8\right)^3}$ (check it!) with line graph:



Therefore the only inflection point is at $x = -\sqrt[3]{16}$. At $x = 0$ we do have $y'' = 0$, but it does not change sign, so it is not an inflection point.

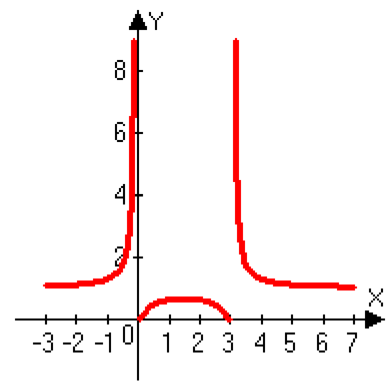
The graph produced by the calculator is as shown here and is consistent with all the information we found, just a little smoother.



The next example shows you a situation where not all features can be identified, but we can still get an informative graph, although incomplete work may hide some important features.

Example: $f(x) = e^{\frac{1}{x^2-3x}}$

This time we are going to cheat and have a peak at the graph from the calculator first! We know that an exponential is always positive, so we set the window accordingly:



It looks like we have two asymptotes and a maximum, with the middle section going … where? It cannot go to below the axis, so does it hit it? And if so, where? And are there other interesting, but invisible features?

Let's see. We start from the intercepts:

$$x=0 \quad \Rightarrow \quad y=DNE \quad ; \quad y=0 \quad \Rightarrow \quad x=DNE$$

Fine, no intercepts. The asymptotes, which are the only two discontinuities, occur at the *x* values for which the exponent becomes infinite, that is, when:

$$x^2-3x=0 \quad \Rightarrow \quad x=0,3$$

Let's check, using brackets to indicate known forms:

$$\lim_{x\to 0^-} e^{\frac{1}{x^2-3x}} = \lim_{x\to 0^-} e^{\frac{1}{x(x-3)}} = \left(e^{\infty}\right) = \infty$$

$$\lim_{x\to 0^+} e^{\frac{1}{x^2-3x}} = \lim_{x\to 0^+} e^{\frac{1}{x(x-3)}} = \left(e^{-\infty}\right) = 0$$

So, this is an asymptote on the left side only, while on the right it is a single point hole! The same happens at the other value:

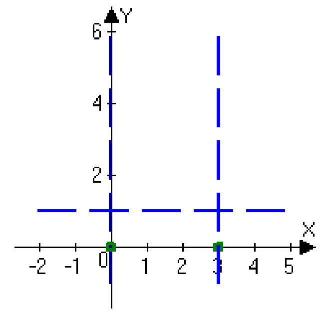
$$\lim_{x\to 3^-} e^{\frac{1}{x^2-3x}} = \lim_{x\to 3^-} e^{\frac{1}{x(x-3)}} = \left(e^{-\infty}\right) = 0$$

$$\lim_{x\to 3^+} e^{\frac{1}{x^2-3x}} = \lim_{x\to 3^+} e^{\frac{1}{x(x-3)}} = \left(e^{\infty}\right) = \infty$$

Now we check for the horizontal asymptotes

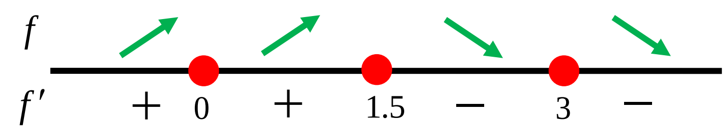
$$\lim_{x\to \infty} e^{\frac{1}{x^2-3x}} = \lim_{x\to -\infty} e^{\frac{1}{x^2-3x}} = e^0 = 1$$

Yes, $y=1$ is a horizontal asymptote. If we had started from scratch, without looking at the calculator, we would sketch this information on the graph:

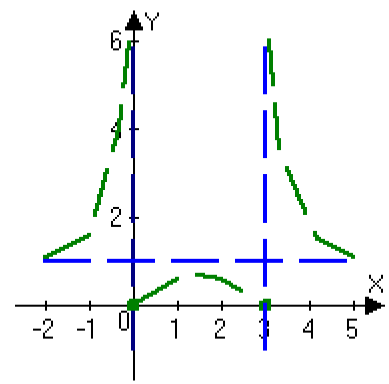


Notice that I have changed the window to highlight the key central features.

The first derivative is $f'(x) = \dfrac{e^{\frac{1}{x^2-3x}}(3-2x)}{\left(x^2-3x\right)^2}$ (check it!) with line graph:



As expected, there is only one critical value and we now know that the maximum is at $x=1.5$. So at this point we can reconstruct a graph very close to what our calculator gave us:



What about the second derivative? It is a bit of a challenge to compute it: you may want to do it during a cold winter night by a roaring fire. But the concavities at this point do not add much, so it may not be worth the effort. Remember that we are just sketching!

However, if you do succeed in obtaining y'', you will get a pleasant surprise: it will reveal two inflection points in the middle section. close to the asymptotes! Not as pleasant as chocolate, but interesting, especially to remind us that the calculator does not always provide all the information, or even accurate hints, eh?

I can see that sketching some of these graphs can become an adventure!

Every graph has its own story and its own surprises. Once you get the hang of it, sketching a graph can be a very fun activity, but only if you learn the key methods, implement them carefully and get plenty of experience. Speaking of which, the *Learning questions* offer several good adventures for you. Keep in mind that you can use your calculator to check your work, but do not trust the calculator all the times: there may be some hidden features and when there is a discrepancy or an unclear aspect, investigate it always!

Summary

- When sketching the graph of a function, we are not aiming for perfection, but only for an accurate representation of the main graphical features of such graph
- The main features we need to obtain are intercepts, discontinuities and behaviour at infinity (from the original function), up/down pattern and extreme points (from the first derivative) and concavity and inflection points (from the second derivative).

Common errors to avoid

- When making a mistake in computation, do not force the features you find to coexist, because they won't! Instead identify the inconsistency and state the need for investigating the problem. Trying to force things will create graphs that are obviously wrong, recognizing errors will show that you are reflecting on what you are finding.

Learning questions for Section D 8-5

Review questions:

1. Describe the three strategies for collecting relevant information about the graph of a function.
2. Describe the strategy for using the information provided by calculus to sketch the graph of a function.
3. Explain why it is impossible to obtain the exact graph of a function.

Memory questions:

1. What kind of information about the graph of a function is obtained by analyzing the original function only?

2. What kind of information about the graph of a function is obtained by analyzing its first derivative?

3. What kind of information about the graph of a function is obtained by analyzing its second derivative?

4. What graphical information is provided by the limits at infinity when the function has no horizontal asymptotes?

Computation questions:

In each of questions 1-18 a function and its first and second derivative are provided. Use them to identify all key graphical features of the function and to sketch an informative graph. Compare your graph to that provided by a graphing calculator, but identify the location of the important features as exactly as you can.
Of course, you may want to compute those derivatives yourself, just to practice and to make sure that there are no errors!

1. $y=\dfrac{1}{x^3-4x}$; $\quad y'=\dfrac{4-3x^2}{(x^3-4x)^2}$; $\quad y''=\dfrac{4(3x^4-6x^2+8)}{(x^3-4x)^3}$

2. $y=\dfrac{2x+1}{x\left(x+1\right)}$; $\quad y'=\dfrac{-\left(2x^2+2x+1\right)}{x^2\left(x+1\right)^2}$; $\quad y''=\dfrac{2\left(2x+1\right)\left(x^2+x+1\right)}{x^3\left(x+1\right)^3}$

3. $y=\dfrac{x^2}{x+8}$; $\quad y'=\dfrac{x^2+16x}{\left(x+8\right)^2}$; $\quad y''=\dfrac{128}{\left(x+8\right)^3}$

4. $y=\dfrac{x+1}{x^2-x}$; $\quad y'=-\dfrac{x^2+1}{\left(x^2-x\right)^2}$; $\quad y''=-2\dfrac{-x^3-2x+1}{\left(x^2-x\right)^3}$

5. $y=\dfrac{x}{e^x}$; $\quad y'=\dfrac{1-x}{e^x}$; $\quad y''=\dfrac{x-2}{e^x}$

6. $y=\dfrac{x}{\ln x}$; $\quad y'=\dfrac{\ln x-1}{(\ln x)^2}$; $\quad y''=\dfrac{2-\ln x}{x(\ln x)^3}$

7. $y=x\ln x^2$; $\quad y'=\ln x^2+2$; $\quad y''=\dfrac{2}{x}$

8. $y=(x-1)^{\frac{2}{3}}-(x+1)^{\frac{2}{3}}$; $\quad y'=\dfrac{2}{3}\left((x-1)^{-\frac{1}{3}}-(x+1)^{-\frac{1}{3}}\right)$; $\quad y''=-\dfrac{2}{9}\left((x-1)^{-\frac{4}{3}}-(x+1)^{-\frac{4}{3}}\right)$

9. $y=\left(x^2-8\right)^{2/3}$; $\quad y'=\dfrac{4x}{3}\left(x^2-8\right)^{-1/3}$; $\quad y''=\dfrac{4}{3}\left(x^2-8\right)^{-\frac{4}{3}}\left(\dfrac{x^2}{3}-8\right)$

10. $y=\dfrac{2x^2}{\left(x+1\right)^2}$; $\quad y'=\dfrac{4x}{\left(x+1\right)^3}$; $\quad y''=\dfrac{4-8x}{\left(x+1\right)^4}$

11. $y=x^2e^{-x}$; $\quad y'=\left(2x-x^2\right)e^{-x}$; $\quad y''=\left(x^2-4x+2\right)e^{-x}$

12. $y=\dfrac{x-1}{\sqrt{x^2+4}}$; $\quad y'=\dfrac{x+4}{\sqrt{\left(x^2+4\right)^3}}$; $\quad y''=\dfrac{4-12x-2x^2}{\sqrt{\left(x^2+4\right)^5}}$

13. $y=x\sqrt{5-x^2}$; $\quad y'=\sqrt{5-x^2}-\dfrac{x^2}{\sqrt{5-x^2}}$; $\quad y''=\dfrac{-2x}{\sqrt{5-x^2}}-\dfrac{10x}{\left(5-x^2\right)^{3/2}}$

14. $y=\dfrac{x-1}{\sqrt{x^2+2}}$; $\quad y'=\dfrac{x+2}{\sqrt{\left(x^2+2\right)^3}}$; $\quad y''=\dfrac{2-6x-2x^2}{\sqrt{\left(x^2+2\right)^5}}$

15. $y=\frac{2}{x(x-3)^2}$; $y'=\frac{-6(x-1)}{x^2(x-3)^3}$; $y''=\frac{12(2x^2-4x+3)}{x^3(x-3)^4}$

16. $y=\frac{e^{x^2}}{x}$; $y'=\frac{(2x^2-1)e^{x^2}}{x^2}$; $y''=\frac{2(2x^4-x^2+1)e^{x^2}}{x^3}$

17. $y=e^{3x-x^2}$; $y'=(3-2x)e^{3x-x^2}$; $y''=(4x^2-12x+7)e^{3x-x^2}$

18. $y=\frac{\ln x}{x^2}$; $y'=\frac{1-2\ln x}{x^3}$; $y''=\frac{6\ln x-5}{x^4}$

19. $y=\frac{x}{\sqrt{x^3+1}}$ $y'(x)=\frac{2-x^3}{2\left(\sqrt{x^3+1}\right)^3}$ $y''(x)=\frac{3}{4}\frac{x^5-5x^2}{\sqrt{\left(x^3+1\right)^5}}$.

20. $y=x+4\sqrt[3]{x^2}$ $y'(x)=1+\frac{8}{3}x^{-\frac{1}{3}}$ $y''(x)=-\frac{8}{3}x^{-\frac{4}{3}}$

For each of the functions provided in questions 21-38, obtain all relevant information and use it to sketch an informative graph of the function.

21. $y=\frac{3}{4}x^4+3x^3-6x^2+10$

22. $f(x)=\frac{x+1}{\sqrt{x^2-2}}$

23. $y=\left(4-x^5\right)^2$

24. $y=5x^3-3x$

25. $y=15x^4-15x^2$

26. $f(x)=\frac{x^3+2}{x^3-8}$

27. $f(x)=2x-\sqrt{x}$

28. $f(x)=\frac{x^3}{x^2-8}$

29. $y = x - 8\sqrt[5]{x^2}$

30. $f(x) = \dfrac{2x}{\sqrt{x^2 - 5}}$

31. $y = \dfrac{x}{2} - \cos x,\ \ 2\pi \le x \le 4\pi$

32. $y = \sin 2x - 2\sin x$

33. $f(x) = \tan x + \cos x,\ \ 0 \le x \le \pi$

34. $y = \sin^{-1}\left(\dfrac{1}{x}\right)$

35. $f(x) = e^{\frac{1}{x^2-4}}$

36. $y = \ln\left(x^2 - 1\right)$

37. $f(x) = x^2 e^{-x^2}$

38. $y = \dfrac{5x(x-3)^4}{(x-1)^3(x+2)^2}$

39. $y = \left(2 - \sinh^2 x\right)^3$ (No need to analyze the second derivative)

Theory questions:

1. Which calculus method is key in the analysis of asymptotes?
2. Mention two graphical features of a function that may not be visible on the calculator
3. At an *x*-value where *y'=DNE*, how do we determine whether we have a vertical asymptote or a vertical tangent line?
4. Identify potential calculator errors that can mislead in identifying the graph of a function.
5. Identify three graphical features that can occur at an $x = c$ if $f'(x) < 0$ when $x < c$ and $f'(x) > 0$ when $x > c$.
6. Can the line tangent to a curve at a point cross the curve at that point?
7. Mention two graphical features that are obtained directly from the original function
8. What features of the function provide information on the window to use when sketching its graph?
9. Why is it necessary to do al the calculus work to sketch the graph of a function, rather than simply using the calculator's graph?
10. If you graph a polynomial function and zoom in repeatedly on any one of its points, what shape will its graph take eventually?

Proof questions:

1. Construct the formula of a function that has vertical asymptote at $x = 1$, a single point hole at $(3, 2)$, and a horizontal asymptote at $y = 4$.

Templated questions:

1. Construct a reasonably simple function and sketch its graph by using calculus methods

What questions do you have for your instructor?