Chapter 1: Indefinite integrals *Section 1*

Antiderivatives and indefinite integrals

What you need to know already:	*What you can learn here:*
➤ How to compute derivatives.	➤ What an antiderivative is. ➤ What an indefinite integral is and what its proper notation is.

Since you know how to go from a function to its derivative, it is now time to investigate a question that is interesting on its own merit, but also leads to unexpected and incredibly fruitful consequences.

The question is how to go backwards, from a derivative to its original function. So far we have computed derivatives and used them for a variety of applications, but is it possible to go backwards? That is, given a function $y = f(x)$, is it possible to find another function whose derivative is *f(x)*? Before we dive into the investigation of this question, here is a basic jargon word.

Definition

Given a function $y = f(x)$, any function $y = F(x)$ such that

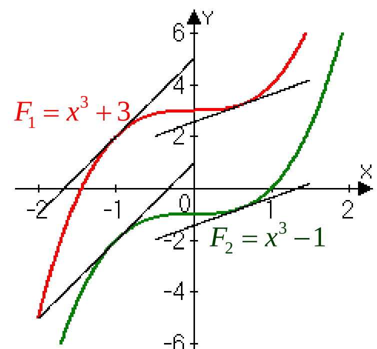
$$F'(x) = f(x)$$

is called an ***antiderivative*** of $f(x)$.

Example: $f(x) = 3x^2$

Since $\left(x^3\right)' = 3x^2$, it follows that $F(x) = x^3$ is an antiderivative of the function $f(x) = 3x^2$.

However, so are $F_1(x) = x^3 + 3$ and $F_2(x) = x^3 - 1$!



That is because the graphs of F_1 and F_2 are the same, except for a vertical shift. Their slope at every value of *x* is the same, and therefore so are their derivatives.

In this last definition I used the common convention of indicating an antiderivative by using the same letter as the original function, but in capital form. So, an antiderivative of $f(x)$ will be denoted by $F(x)$, an antiderivative of

$g(x)$ by $G(x)$, and so on. Notice that one could also use the notation $f^{(-1)}(x)$ for an antiderivative, since this is consistent with the notation for higher derivatives. But since this notation can easily be confused with the one for inverse and reciprocal functions, it is seldom used.

But I have seen antiderivatives denoted as $\int f(x)\,dx$.

You are right, but this notation has a slightly different meaning and even a different name. I'll get to it in a page!

Since the derivative of an additive constant is 0, it follows that given any antiderivative $y = F(x)$ of $y = f(x)$, we can get infinitely many more antiderivatives by considering functions of the form $y = F(x) + c$, where c is any constant. This leads to another name.

Definition

If $y = f(x)$ is a function and $y = F(x)$ is any of its antiderivatives, the ***family of functions***:

$$y = F(x) + c$$

is called a ***general antiderivative*** of $y = f(x)$

Example: $f(x) = 3x^2$

We have seen that $F(x) = x^3$ is one antiderivative of $y = 3x^2$, so the family of all functions of the form $F_c(x) = x^3 + c$ is its general antiderivative.

But we can even go further than that! If $y = f(x)$ has some discontinuities, we can construct new antiderivatives by attaching several constants, one for each unbroken piece of the domain of the function.

Example: $f(x) = -\dfrac{1}{x^2}$

We can recognize this as being the derivative of $y = \dfrac{1}{x}$, so that $F(x) = \dfrac{1}{x}$ is one antiderivative and $F_c(x) = \dfrac{1}{x} + c$ is its general antiderivative. But the function:

$$y = \begin{cases} \dfrac{1}{x} + 3 & \text{if } x < 0 \\ \dfrac{1}{x} - 5 & \text{if } x > 0 \end{cases}$$

is also an antiderivative, since it is undefined at $x = 0$ and its derivative on each piece is our original function.

This fact leads to the definition and notation you mentioned, a notation that will be used consistently throughout the rest of the course.

Definition

If $y = f(x)$ is a function that has one antiderivative, the set of ALL of its antiderivatives is called the ***indefinite integral*** of $f(x)$, and is denoted by:

$$\int f(x)\,dx$$

You have probably seen this notation before and may be somewhat familiar with its origin and the meaning of its parts, but we shall see them all in detail in later sections.

Example: $f(x)=3x^2$

We have seen that $F(x)=x^3$ is one antiderivative of $y=3x^2$, so the family of all of its antiderivatives is written as $F(x)=x^3+c$.

But isn't this the same as the general antiderivative?

Some people equate the two concepts, and that is fine. I prefer to make the distinction between a general antiderivative, of the form $F_c(x)=x^3+c$, and the indefinite integral, indicated by $\int 3x^2dx$, that includes ALL antiderivatives. Now, when the original function is continuous the two concepts coincide.

Technical fact

If $f(x)$ is a continuous function and $F(x)$ is one of its antiderivatives, then:

$$\int f(x)dx=F(x)+c$$

where c can be any real number.

Proof

If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+c$, since:

$$(F(x)+c)'=(F(x))'+0=f(x).$$

If $G(x)$ is another antiderivative and $f(x)$ is continuous, then $F(x)$ and $G(x)$ must both be continuous, since differentiability implies continuity. Therefore, their difference $G(x)-F(x)$ is a continuous function whose derivative is 0. The only such functions are constant (see if you can show it!), so that:

$$G(x)-F(x)=c \quad\Rightarrow\quad G(x)=F(x)+c$$

Example: $f(x)=\cos x$

We know that $f(x)=\cos x$ is the derivative of $F(x)=\sin x$ and that it is continuous. Therefore, $\int \cos x\,dx=\sin x+c$

So, a difference occurs only when there are discontinuities, then!

Right: when the function to integrate has some discontinuities, the additive constants required determines the distinction between general antiderivative and indefinite integral. At least for sticklers like me! ☺

Technical fact

When the function $f(x)$ has ***discontinuities***, in order to find its indefinite integral we need to:

1. ***Divide its domain*** into intervals over each of which the function is continuous.

2. Determine the general antiderivative ***for each such interval***.

3. Write the general antiderivative as a ***piece-wise function***, with one formula and a ***different constant for each such interval***.

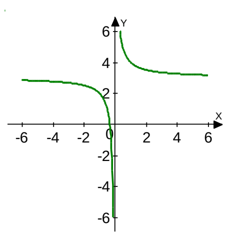
Example: $f(x) = -\dfrac{1}{x^2}$

We have seen that the general antiderivative of $f(x) = -\dfrac{1}{x^2}$ is $F_c(x) = \dfrac{1}{x} + c$. However, $f(x)$ is discontinuous at $x = 0$, so to get the whole indefinite integral, we need to consider the two intervals $x < 0$ and $x > 0$ separately and attach a different constant to each of them. In both cases the known antiderivative is the same, so we should write:

$$\int \left(-\frac{1}{x^2}\right) dx = \begin{cases} \dfrac{1}{x} + c_1 & if \ x < 0 \\ \dfrac{1}{x} + c_2 & if \ x > 0 \end{cases}$$

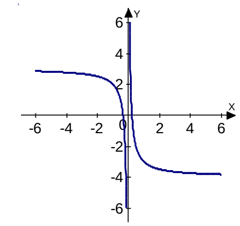
For instance, notice that:

$$F_3(x) = \begin{cases} \dfrac{1}{x} + 3 & if \ x < 0 \\ \dfrac{1}{x} + 3 & if \ x > 0 \end{cases}$$



is an antiderivative of our original function, but so is the function:

$$G(x) = \begin{cases} \dfrac{1}{x} + 3 & if \ x < 0 \\ \dfrac{1}{x} - 4 & if \ x > 0 \end{cases}$$



for which the two constants are different.

Boy, that is a subtle difference!

I agree, and I will not insist too much on it. In fact, in order to keep the notation simple, I will usually write $\int f(x)\,dx = F(x) + c$ even in situations where several constants are warranted. That is:

Knot on your finger

The expressions "*general antiderivative*" and "*indefinite integral*" will be ***used interchangeably***, even when they refer to different families of functions, ***unless the distinction is important*** for some specific purpose.

But you do need to be aware of this subtle difference for the occasional cases where the difference matters.

Notice that the construction of antiderivatives and indefinite integrals is an ***inverse*** process, just like division is the inverse of multiplication and taking roots is the inverse of taking powers. And just like for these more familiar operations, finding antiderivatives is a more complex process than finding derivatives and in some situations cannot be done at all (like dividing by 0 or taking the root of a negative number).

Also, like other inverse operations, the simplest way to start learning how to compute antiderivatives is by reversing differentiation formulae. You can start doing that through the learning questions and I will offer you even more opportunities after introducing some more notation and terminology.

Summary

- A function $y = F(x)$ whose derivative is $y' = f(x)$ is called an *antiderivative* of $y = f(x)$.
- By adding a constant to an antiderivative, we obtain the *general antiderivative*.
- The set, or family, consisting of all antiderivatives of a given function is called its *indefinite integral* and denoted by $\int f(x)\,dx$.

Common errors to avoid

- Don't forget that when a function is discontinuous, its indefinite integral must exhibit a separate constant for each continuous piece.

Learning questions for Section I 1-1

Review questions:

1. Describe what an antiderivative is.
2. Describe what the general antiderivative is.
3. Describe what an indefinite integral is.
4. Compare and contrast the concepts of antiderivative, general antiderivative and indefinite integral.

Memory questions:

1. What is the symbol for the indefinite integral of a function $f(x)$?
2. How many constants are needed for the indefinite integral of a function?

Computation questions:

1. Check whether $F(x)=\cos x+x^2+3$ is an antiderivative of $f(x)=\sin x+2x$.

2. Check whether $G(x)=\ln x$ is an antiderivative for $g(x)=\dfrac{1}{x}$.

Determine the general antiderivative and, if different, the indefinite integral of the function provided in questions 3-8.

3. $y=3$

4. $y=e^x$

5. $y=\sinh x$

6. $f(x)=\sec^2 x$.

7. $f(x)=\cos x-\dfrac{1}{\sqrt{1-x^2}}$.

8. $y'=\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{1+x^2}$

9. Compute the general antiderivative of the function $f(x)=\begin{cases}\cos 2x & if\ \ x\le 0\\ e^{2x} & if\ \ x\ge 0\end{cases}$

Theory questions:

1. Do all continuous functions have an antiderivative?

2. The power rule for derivatives works for any exponent. For which exponents is the same true when computing antiderivatives?

3. Determine $\int\left(\sinh^{-1}3x^2\right)'\,dx$

4. Is it true that if $f(x)$ is a differentiable function, then $\left(\int f(x)dx\right)' = f(x)+c$?

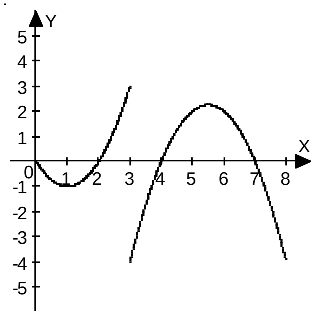
5. If $f'(x)=g'(x)$, how are $f(x)$ and $g(x)$ related?

6. If a function *f(x)* is discontinuous at a single point, say *x*=2, does it follow that all its antiderivatives are discontinuous at that point as well?

7. How can one check if one has found the correct antiderivative of a given function?

Proof questions:

1. The rate of change of a certain quantity over time is shown in this graph.



Sketch a graph that may represent the value of the quantity over the same period of time. Explain the reasons for the main features of your graph.

2. Prove that $\int \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \right) dx = x\sqrt{1-x^2} + c$ and determine if the right side is the general antiderivative, the indefinite integral or both.

Application questions:

1. Which function $f(x)$ is such that $f'(x) = 2x^3 + 1$ and the line $y - 3x = 1$ is tangent to it?

2. Problem: An airplane starts its take off run from rest and with constant acceleration. After covering 1 km it takes off at a speed of 240 km/hr. What is the plane's acceleration?

3. A car can accelerate from 25 to 80 km/hr in 12 seconds. Assuming a constant acceleration, how far does the car travel during those 12 seconds? What if the acceleration is increasing linearly? In this case, is the above information sufficient?

Templated questions:

1. For any indefinite integral you compute, determine how many constants are technically needed.

What questions do you have for your instructor?