Chapter 1: Indefinite integrals — *Section 3*

Basic integration formulae

What you need to know already:	**What you can learn here:**
➤ What an indefinite integral is and what its notation and terminology are.	➤ How to compute indefinite integrals in simple cases.

Computing general antiderivatives (another name for indefinite integrals, remember?) is an *inverse* operation, just like division is the inverse of multiplication and taking roots is the inverse of taking powers. Therefore, in general, integration tends to be a more difficult and complicated process than differentiation. In fact, in some situations it cannot be done at all, just like dividing by 0 and square rooting a negative number cannot be done.

Also, as for other inverse operations, the simplest way to start learning how to do it is by reversing the original operation in particularly easy cases. For integration, this means reversing basic differentiation formulae. So, here are the simplest integration formulae, obtained by reading the corresponding differentiation rules.

Technical fact:

Basic integration formulae

Constant rule: $\int a\, dx = ax + c$

Coefficient rule: $\int a \cdot f(x)\, dx = a\int f(x)\, dx$

Addition rule: $\int [f \pm g]\, dx = \int f\, dx \pm \int g\, dx$

Power rule: $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c,\ n \neq -1$

Exponential rule: $\int e^x\, dx = e^x + c$

Reciprocal rule: $\int x^{-1}\, dx = \ln|x| + c$

Trig rules: $\int \cos x\, dx = \sin x + c$

$\int \sin x\, dx = -\cos x + c$

$\int \sec^2 x\, dx = \tan x + c$

Inverse trig rules: $\int \dfrac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$

$\int \dfrac{1}{x^2+1}\, dx = \tan^{-1} x + c$

Hyperbolic rules: $\int \cosh x\, dx = \sinh x + c$

$$\int \sinh x\, dx = \cosh x + c$$

Inverse hyperbolic rules:

$$\int \frac{1}{\sqrt{x^2+1}}\, dx = \sinh^{-1} x + c$$

$$\int \frac{1}{\sqrt{x^2-1}}\, dx = \cosh^{-1} x + c$$

$$\int \frac{1}{1-x^2}\, dx = \tanh^{-1} x + c, \ \ if\ -1 < x < 1$$

Proof

Most of these formulae follow immediately from the corresponding differentiation formulae and you can check them on your own. The only two that need a little attention are the formulae to integrate a power.

Since $\left(x^n\right)' = nx^{n-1}$, as long as $n \neq 0$, the power rule tells us that for any non-zero power of *x*, the derivative is obtained by:

1. multiplying the power by the exponent, then
2. decreasing the exponent by 1.

Therefore, in order to invert the process we need to:

1. increase the exponent by 1
2. divide the power by the new exponent.

This works, as long as the new power is not the old exception, that is, as long as $n+1 \neq 0$. This leads to the formula $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \ \ n \neq -1$.

What about the exceptional value? What is the integral of the reciprocal function?

When dealing with derivatives we learned that $\left(\ln x\right)' = \frac{1}{x} = x^{-1}$, but this is valid only for positive values of *x*, since that is the domain of the natural logarithm. But we can use the chain rule to see that:

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{d}{dx}\left(\ln\sqrt{x^2}\right) = \frac{1}{\sqrt{x^2}}\frac{1}{2\sqrt{x^2}}2x = \frac{x}{x^2} = \frac{1}{x}$$

Since this time the original function is defined for all non-zero values of *x*, the formula $\int x^{-1}\, dx = \ln|x| + c$ is valid.

Before showing you some examples, let me remind you of a technical fact that will not have any major impact in this course, but will become important in more advanced uses of integrals.

Knot on your finger

The formulae listed in the previous table provide the general antiderivative of the corresponding integrand.

To obtain the indefinite integral we need to consider a different constant on each interval of continuity.

Can you identify the formulae to which this technical difference applies?

Example: $\int \left(\cos x - \frac{3}{x^2} + \frac{4}{x}\right) dx$

To compute this integral, we first apply the addition rule:

$$\int\left(\cos x-\frac{3}{x^2}+\frac{4}{x}\right)dx=\int\cos x\,dx-\int\frac{3}{x^2}dx+\int\frac{4}{x}dx$$

Notice how the differential is repeated in each new integral: it cannot be absent from any of them! Now we use the coefficient rule, thus taking the coefficients out of the last two integrals:

$$=\int\cos x\,dx-3\int x^{-2}dx+4\int x^{-1}dx$$

For the first integral we can now use the basic trig rules, on the second the power rule, while on the third we use the reciprocal rule:

$$=\sin x-3\frac{x^{-2+1}}{-2+1}+4\ln|x|+c$$

If there is a next step in the problem at hand, we may want to re-write this expression as:

$$\int\left(\cos x-\frac{3}{x^2}+\frac{4}{x}\right)dx=\sin x+\frac{3}{x}+4\ln|x|+c$$

We could stop here, but to be precise, we notice that the original function is discontinuous at $x=0$, so that what we have so far is the general antiderivative. The technically correct indefinite integral is:

$$\int\left(\cos x-\frac{3}{x^2}+\frac{4}{x}\right)dx=\begin{cases}\sin x+\dfrac{3}{x}+4\ln|x|+c_1 & \text{if } x<0\\ \sin x+\dfrac{3}{x}+4\ln|x|+c_2 & \text{if } x>0\end{cases}$$

The next example reminds you that we are still at the very beginning of our work with integrals!

Example: $\int\left(\cos x^2-\frac{3}{x^2-2}+\frac{xe^x}{4}\right)dx$

To compute this integral we can still use the addition and coefficient formulae:

$$\int\left(\cos x^2-\frac{3}{x^2-2}+\frac{xe^x}{4}\right)dx=$$

$$=\int\cos x^2dx-3\int\frac{1}{x^2-2}dx+\frac{1}{4}\int xe^xdx$$

But now we must stop: we do not have formulae for any of these three functions! We have formulae that look *similar* to them, but we can only use what we *know to be true*. For similar cases we need to verify that the formula still holds, and in none of the three remaining integrals this is true. You may want to check by differentiating that what you believe to be the correct antiderivative based on the similar formulae you do NOT get the original integrand. It's a great exercise!

When the integrand consists of a basic derivative composed with a linear function, we may guess and check its antiderivative by using the simple outcome of the linear rule.

What does that mean?

Example: $\int\cos(2-3x)\,dx$

This integrand consists of a function whose antiderivative we know (the cosine) composed with a linear function whose derivative is simply the constant -3. The chain rule would produce that constant as a coefficient, but we can take care of it by suitably multiplying and dividing:

$$\int\cos(2-3x)\,dx=\frac{-1}{3}\int(-3)\cos(2-3x)\,dx$$

Now we recognize the integrand as the derivative of a function we know, so that we can conclude that:

$$\int\cos(2-3x)\,dx=\frac{-1}{3}\sin(2-3x)+c$$

That is neat! Is that an example of a general rule?

Well, it is an example of a general method that we shall see in detail in the next section. For now, try using it in similarly simple cases and look forward to the splendid method from which it comes!

Now that we have a few basic integration rules, we can use them in applied problems. Here is a basic one.

Problem:

Suppose that a bowling ball is released at the player's end with a speed of $6\frac{m}{s}$ *and, due to the friction, it is subject to an deceleration given by*

$$a(t) = \frac{\cosh t}{50} - \frac{1}{2} \quad \frac{m}{s^2}$$

What is the exact distance travelled by the ball in the first 2 seconds?

Solution:

Since we know the acceleration, we can compute the velocity:

$$v(t) = \int \left(\frac{\cosh t}{50} - \frac{1}{2} \right) dt = \frac{\sinh t}{50} - \frac{t}{2} + c$$

Since the initial velocity, that is, at time $t = 0$, is 6, we have:

$$\frac{\sinh 0}{50} - \frac{0}{2} + c = 6 \quad \Rightarrow \quad c = 6 \quad \Rightarrow \quad v(t) = \frac{\sinh t}{50} - \frac{t}{2} + 6$$

Now that we have the velocity, we can find the position:

$$s(t) = \int \left(\frac{\sinh t}{50} - \frac{t}{2} + 6 \right) dt = \frac{\cosh t}{50} - \frac{t^2}{4} + 6t + c$$

Since the initial position of the ball is 0, we have:

$$\frac{\cosh 0}{50} - \frac{0^2}{4} + 0 + c = 0 \quad \Rightarrow \quad c = -\frac{1}{50}$$

$$\Rightarrow \quad s(t) = \frac{\cosh t}{50} - \frac{t^2}{4} + 6t - \frac{1}{50}$$

Therefore after two seconds the ball has travelled:

$$s(2) = \frac{\cosh 2}{50} - 1 + 12 - \frac{1}{50} \, m$$

In future sections, we shall see many more applications of antiderivatives. In fact, integration is a much more applied method than differentiation. By differentiating, we compute a rate of change and all we can do to extend its use is give it different interpretations. But with integration we can compute quantities that at first sight look substantially different, such as forces and volumes. Isn't that interesting? Stay tuned for more fun on this.

Summary

- Basic integration formulae are obtained by reversing simple differentiation formulae.

Common errors to avoid

- Just because an integrand looks *similar* to one whose antiderivative you know, it *does NOT follow* that the true antiderivative is similar to what you expect!
- Apply an integration formula *ONLY* if it applies *exactly* to the integrand you are dealing with.

Learning questions for Section I 1-3

Review questions:

1. Describe how the basic integration formulae are obtained.

2. Explain why there are two different integration rules for power functions.

Memory questions:

1. What is the general antiderivative of $f(x) = x^n$ if $n \neq -1$?

2. What is the general antiderivative of $f(x) = \frac{1}{x}$?

3. What is the general antiderivative of $f(x) = e^x$?

4. What is the general antiderivative of $f(x) = \sin x$?

5. What is the general antiderivative of $f(x) = \cos x$?

6. What is the general antiderivative of $f(x) = \sec^2 x$?

7. What is the general antiderivative of $f(x) = \frac{1}{\sqrt{1-x^2}}$?

8. What is the general antiderivative of $f(x) = \frac{1}{1+x^2}$?

9. What is the general antiderivative of $f(x) = \sinh x$?

10. What is the general antiderivative of $f(x) = \cosh x$?

11. What is the general antiderivative of $f(x) = \frac{1}{\sqrt{1+x^2}}$?

12. What is the general antiderivative of $f(x) = \frac{1}{\sqrt{x^2-1}}$?

13. What is the general antiderivative of $f(x) = \frac{1}{1-x^2}$?

Computation questions:

1. Compute $\int\left(\frac{1}{\sqrt{1-x^2}}-\frac{1}{1+x^2}\right)dx$.

2. Determine $\int\left(\cos x-\frac{1}{\sqrt{1-x^2}}\right)dx$.

3. Compute $\int\frac{3}{4x^2+4}\,dx$.

4. What is the general antiderivative of the function $y=5^x$?

5. Find a function $y=f\left(x\right)$ such that $y'+3=2\left(1-x^2\right)^{-\frac{1}{2}}$, $y\left(1\right)=2$.

6. Which function $f\left(x\right)$ is such that $f'\left(x\right)=\frac{4+3x^2}{\sqrt{x}}$ and $f\left(4\right)=2$?

7. Determine the function $f\left(x\right)$ whose derivative is $y=3\sin x+e^x$ and whose y-intercept is at $\left(0,3\right)$.

8. Determine the function $f\left(x\right)$ whose derivative is $y=e^x-\cos x$ and whose y-intercept is at $\left(0,5\right)$.

9. Which function $\beta\left(\sigma\right)$ is such that $\beta'\left(\sigma\right)=\sin 2\sigma+e^{\sigma}$ and $\beta\left(0\right)=1$?

10. Find a function that belongs to the indefinite integral $\int\left(\frac{1}{\sqrt[3]{x}}-4^x\right)dx$ and is such that $f\left(1\right)=2$.

11. Compute $\int\frac{1}{x-1}dx$.

12. Determine $\int\frac{1}{\left(x-1\right)^2}dx$.

13. Find the general antiderivative of the function $y=\frac{e^{-x}}{3}-\frac{4}{1+x^2}$

14. Compute the general antiderivative of $y=e^x+\frac{3}{\sqrt{1-x^2}}$

15. Find all functions $y=f\left(x\right)$ for which $y''=\cos 2x+\frac{2}{x^2}$

16. Determine a function $y=f\left(x\right)$ whose derivative is $y'=5\sinh x-\frac{3}{1-x^2}$ and whose graph contains the point $\left(0,3\right)$.

17. Find a function $y=f\left(x\right)$ such that $y''+\sqrt[3]{x}-1=0$, $y\left(0\right)=1$ and $y'\left(1\right)=2$

18. Compute $\int \left(\cos 2x + \frac{3x^2 - 1}{x^3} \right) dx$.

Theory questions:

1. Does the formula $\int x^{-1}\, dx = \ln|x| + c$ express the general antiderivative, the indefinite integral or both?

2. The list of basic integration formulae does not involve the hyperbolic tangent and its inverse. Why do you think that is?

3. Is it true that $\int \left[af(x) + bg(x) \right] dx = a\int f(x)dx + b\int g(x)dx$?

4. For which value of the exponent is the power rule for integrals NOT valid?

Application questions:

1. Two balls are thrown vertically up at the same time with initial speeds of 15m/s and 10m/s. Use antiderivatives to construct their position functions and use these functions to determine how long the first ball will remain in the air after the second will hit the ground.

2. An object is dropped through a fluid, so that its acceleration is given by the function $a(t) = -\frac{9.8}{t^2 + 1} + 5$. Which formula represents the velocity of the object?

3. A drag racing car begins a race by traveling for 5 seconds with a constant acceleration of 15 m/sec^2, after which a parachute slows it down for 4 seconds to a speed of 25 m/sec, before being detached by the lack of further deceleration. If the car's deceleration decreases linearly over time, how far will the car travel from the start until the parachute is detached?

4. Prove that if an object moves on a straight line with constant acceleration (for instance, a falling object with no air resistance) the average velocity between any two times equals the average of the velocities at those same times. This is a famous fact proved by Galileo and that arguably provided the motivation for Newton's general work on calculus.

Templated questions:

1. For any indefinite integral you compute, determine how many constants are technically needed.

2. In your answer to any computation or application question, clearly identify which formulae of integration are needed, at the step where they are needed.

What questions do you have for your instructor?