The area problem

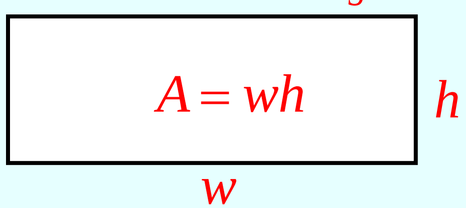
What you need to know already:	What you can learn here:
➤ Basic geometry concepts related to area.	➤ What the *area problem* is in general.

Measuring the area of a region is a very old and very practical problem, arising from diverse applications in agriculture, architecture, engineering, fashion etc. To even ask questions about areas we need to define what it means. To some extent, this is an axiomatic issue (if you don't know what this means, ask me!), so I will assume that we know what a rectangle is and set the following definition:

Definition

The area of a ***rectangle*** is defined as:

$$Area = width \times height$$



Notice that since this is a definition, it does not require a proof. However, the following easy consequences of the definition do require a proof. I said that they are *easy* consequences, and in fact they have been known from time immemorial. So I defer such proofs to you in the *Learning questions*.

Technical facts

The area of a ***square*** with side of length ***s*** is given by:

$$A = s^2$$

The area of a ***triangle*** with base of length ***b*** and height ***h*** is given by:

$$A = \frac{bh}{2}$$

The area of a ***polygonal*** region is obtained by dividing it into many small triangles and adding up the areas of all such triangles.

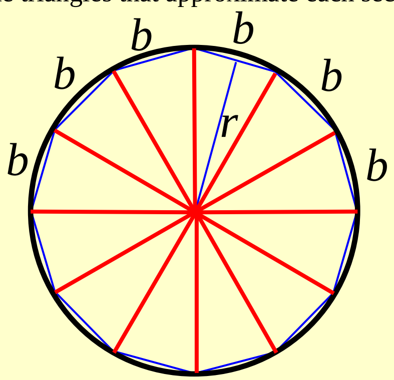
But what about regions with *curved* boundaries? The most appealing such region is a circle and even good old Archimedes found a way to convince everyone that the area of a circle is one half of the product of the circumference and the radius, from which the familiar formula follows:

$$A = \frac{Cr}{2} = \frac{2\pi rr}{2} = \pi r^2$$

Here is the gist of his argument, an argument you should know both for its historical significance and for another reason I will mention right after.

Quick portrait of Archimedes argument for the area of a circle

Consider a circle of radius *r* and therefore with a circumference $C = 2\pi r$. If we divide it into *n* sectors of the same size, as shown in the picture, its area is well approximated by the sum of the areas of the triangles that approximate each sector.



The base of each triangle is approximately equal to the length of the arc that closes the sector, namely $b = C/n$, while the height of each triangle is approximately equal to the radius of the circle, so we can write:

$$A \approx n\left(\frac{1}{2}br\right) = \frac{1}{2}r(nb) = \frac{1}{2}rC = \frac{1}{2}r(2\pi r) = \pi r^2$$

Archimedes' argument then was that as we divide the circle in a larger and larger number of sectors, the approximation becomes better and better, since the unaccounted areas become smaller and smaller. Therefore, if we *exhaust* all possibilities for the pieces of the circle, by using an infinite number of sectors, the approximation will become an equality.

For this reason, this is called an argument by exhaustion: it is supposed to exhaust all possibilities, not you!

I am not calling Archimedes' argument a "*proof*" for technical reasons of formality related to what constitutes a proof. But notice how close Archimedes came to the concept of limit, discovered by Newton and Leibniz only 1800 years later. So, don't feel bad if you are still struggling with that same concept!

It is that concept of limit that we shall use to generalize Archimedes' idea and develop a method to compute the area of any finite plane region.

Definition

The ***area problem*** is the problem of finding a general method to compute the area of any finite region whose boundary is known.

Solving the area problem is what this chapter will be all about.

Summary

- The area problem is the general problem of finding a way to compute the area of any finite plane region, regardless of the shape of its boundary.
- When the boundary is straight, the problem can be solved entirely by using the definition of area of a rectangle – which does not require a proof – and applying some simple geometrical arguments.
- The problem was solved in antiquity for certain special regions, such as a circle, but it had to wait until the invention of calculus to be solved more generally.

Common errors to avoid

- Don't underestimate the importance of knowing the formulae for the area of basic regions and to understand why they are that way.

Learning questions for Section I 4-1

Review questions:

1. Describe, in your own words, what the area problem is.
2. List the basic geometric shapes whose areas you know how to compute.
3. List the basic geometric shapes whose areas you do NOT know how to compute and for each of them find out if there is a known formula and, if so, what it is.

Memory questions:

1. Which formula provides the area of a rectangle?
2. Which formula provides the area of a square?
3. Which formula provides the area of a triangle?
4. Which formula provides the area of a trapezoid?
5. Which formula provides the area of a circle?

Theory questions:

1. Which area formula requires no proof?

Proof questions:

1. Use the definition of area to prove the formula for the area of a square.
2. Use the definition of area to prove the formula for the area of a triangle.
3. Construct the formula for the area of a trapezoid.
4. Construct the formula for the area of a circular sector.
5. Construct the formula for the area of a hexagon.

What questions do you have for your instructor?