Chapter 4: Definite integrals and the FTC — *Section 2*

Riemann sums

What you need to know already:	*What you can learn here:*
➤ The definition of area for a rectangle.	➤ The key method for approximating the area of a curved region, which will lead to the method for computing it exactly.

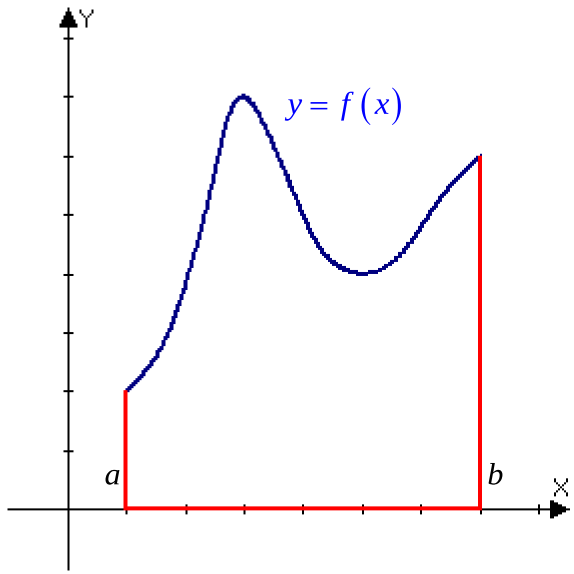
Remember that our current problem is how to compute the area of a plane region with a curved boundary. We have seen that this problem had been solved in antiquity for particular cases, such as the circle. But to achieve our goal we have to move towards a more general situation. We shall begin to do so by considering regions that are almost rectangles, except for one curved side. Here is a more formal description of what we shall analyze.

Definition

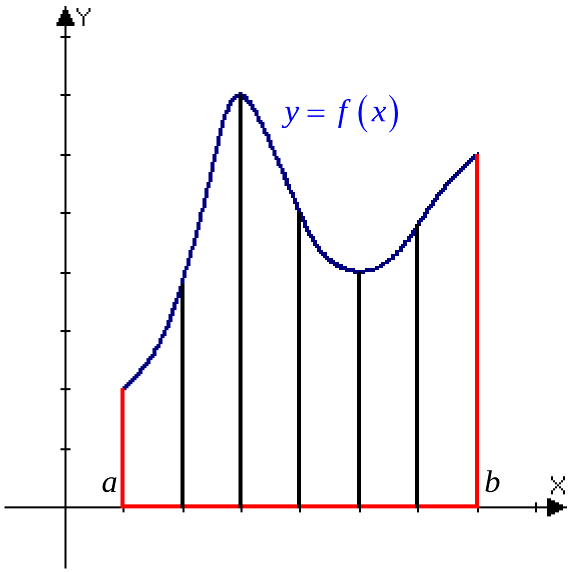
By the ***area under a curve***, we mean the area of the region bounded:

- by the x-axis ***below***
- by the vertical line $x = a$ to the ***left*** and the vertical line $x = b$ to the ***right***, with $a < b$
- by the positive and continuous function $y = f(x)$ ***above***.

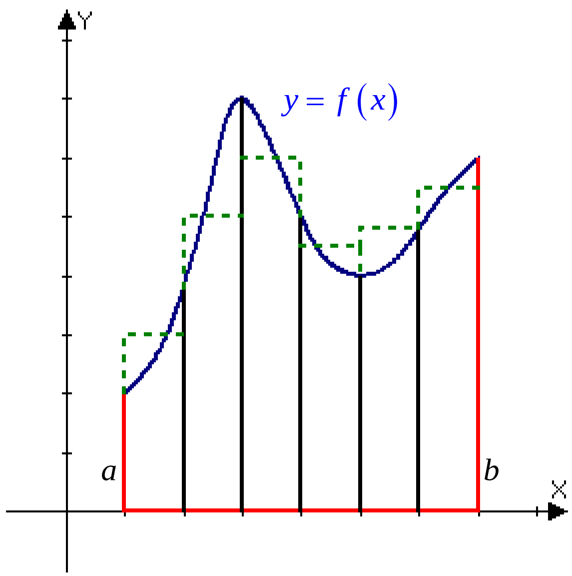
Such a region will look like this:



To approximate its area, we use Archimedes' idea, but instead of dividing the region into sectors, as he did for the circle, we divide it into vertical slices.



We then approximate the area of each slice by using rectangles.

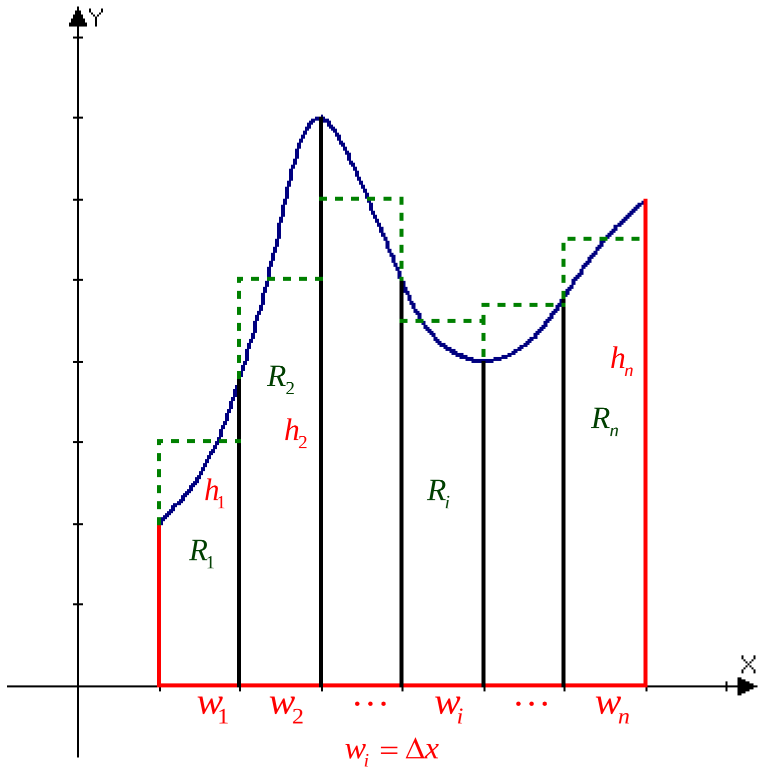


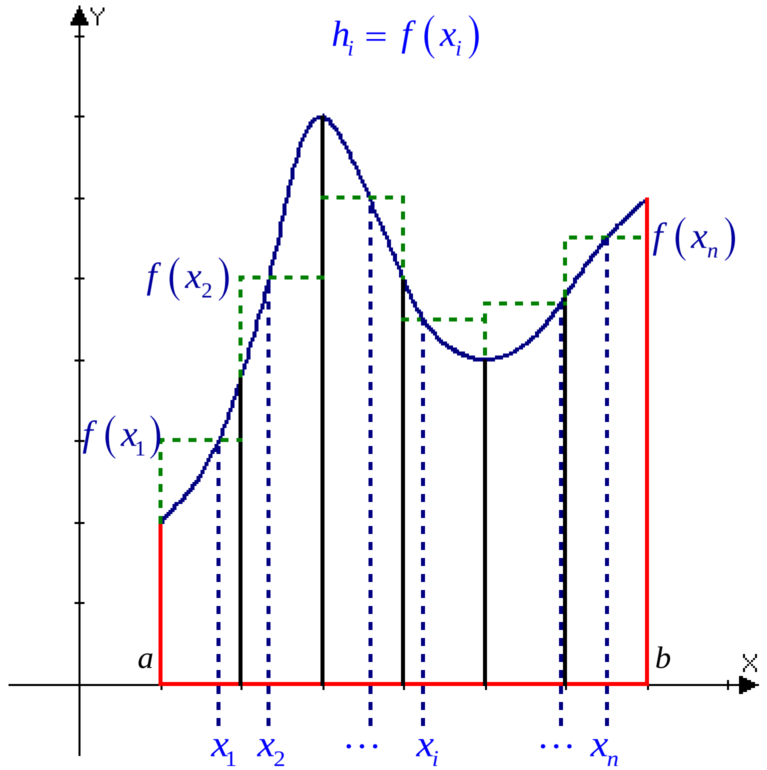
Notice that, unlike what happened to Archimedes with the circle, here it is not clear what height we should assign to each rectangle so as to have a reasonable approximation. We have several choices, some of which will be identified specifically in the next section. For now, we shall focus on organizing the notation, so as to be able to work with it efficiently.

Notation for the approximation of the area under a curve

- We label the approximating rectangles from the left to the right as $R_1, R_2, ..., R_n$, each having width w_i and height $h_i, i = 1, 2, ..., n$.
- For convenience we shall divide the interval $[a, b]$ into n intervals of equal length, so that $w_i = \frac{b-a}{n} = \Delta x$ for all rectangles.
- The ***height*** of the i-th approximating rectangle is $f(x_i)$ for some value x_i in the i-th interval.
- The ***area*** of the i-th approximating rectangle is therefore represented by $A_i = w_i f(x_i)$, or, with equal widths, $A_i = f(x_i)\Delta x$.
- The ***sum of the areas*** of the n approximating rectangles is given by $A = \sum_{i=1}^{n} f(x_i) w_i$, or, with equal widths, $A = \sum_{i=1}^{n} f(x_i)\Delta x$.

This notation is demonstrated in these pictures.





The formula that we shall use to approximate the whole area is a key piece of information that will be used repeatedly. It is very important and it has its own name.

Definition

A quantity of the form:

$$\sum_{i=1}^{n} f(x_i)\Delta x_i$$

is called a ***Riemann sum*** of the function $y = f(x)$ on the interval $[a, b]$ and it provides an approximation to the area under that function on the same interval.

In particular, when using intervals of equal width, the formula:

$$\sum_{i=1}^{n} f(x_i)\Delta x, \ \Delta x = \frac{b-a}{n}$$

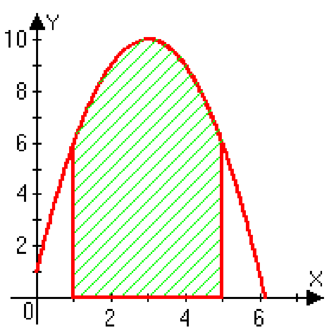
provides a Riemann sum for the function.

Example: $f(x) = 1 + 6x - x^2, \ a = 1, \ b = 5$

We can construct a Riemann sum for this region by using $n = 4$ and intervals of equal widths: $[1, 2], [2, 3], [3, 4], [4, 5]$. This gives:

$$\sum_{i=1}^{4} f(x_i)\Delta x =$$

$$= \sum_{i=1}^{4}\left(1 + 6x_i - x_i^2\right)\frac{5-1}{4}$$



Within each interval we pick a value for x_i say $x_i = 1.2, 2, 3.5, 4.6$. Notice that we can pick these values as we wish!

With these choices, the Riemann sum we obtain is:

$$= \left(1+6(1.2)-1.2^2\right)+\left(1+6(2)-2^2\right)+\left(1+6(3.5)-3.5^2\right)+\left(1+6(4.6)-4.6^2\right)$$

Its value is 32.95. Boy, did that take a lot of steps, and for an easy case! But we'll soon develop a better way ☺.

Notice that a Riemann sum accomplishes the goal of *approximating* the area under the curve and it does so in a way that allows for better and better approximations simply by using a greater number of slices, all with smaller widths. But we have not solved the area problem, since we still have some issues to resolve:

1) Since we need to make a choice of strips and representative points, we end up with different approximations for each such choice, so we need to assess the properties of such possible options.
2) What we really want is the exact value of such area, not just an approximation.
3) The notation we are building is very cumbersome and needs to be refined.

Starting from the next section, we shall address all these issues and come up, thanks to Newton, Leibniz, Riemann and many other mathematicians, with a very workable solution to the general problem. I shall conclude this section by mentioning some names associated with special choices of the values in each interval.

Definition

If we denote by x_i the value we choose in the *i*-th interval, then:

- A ***left Riemann sum*** is obtained by choosing x_i to be the left end point of the *i*-th interval.
- A ***right Riemann sum*** is obtained by choosing x_i to be the right end point of the *i*-th interval.
- A ***midpoint Riemann sum*** is obtained by

choosing x_i to be the midpoint of the *i*-th interval.

- An ***upper Riemann sum*** is obtained by choosing x_i to be the point of the *i*-th interval for which $f(x)$ is largest.
- A ***lower Riemann sum*** is obtained by choosing x_i to be the point of the *i*-th interval for which $f(x)$ is smallest.

Although this terminology is used frequently when working with Riemann sums, we shall not use it much, since we will soon leave Riemann sums behind us.

Summary

- The area under a curve can be well approximated by a Riemann sum consisting of the area of thin rectangles, each approximating a slice of the region in question.
- There are several possible Riemann sums, each linked to a particular choice of how the slices are constructed.

Common errors to avoid

- Pay attention to the notation we are using, since it will become the most important issue in what comes later.

Learning questions for Section I 4-2

Review questions:

1. Describe what a Riemann sum for a function over an interval is.

2. Explain the role of the Riemann sum in the process of solving the area problem.

Memory questions:

1. What basic geometric shape is used when setting up the Riemann sum formula?

2. Which formula represents a Riemann sum?

Computation questions:

1. Construct a Riemann sum that approximates the area under the curve $y = \sin x$ on $[0, \pi]$ by using 6 slices.

2. Construct a Riemann sum that approximates the area under the curve $y = \ln x$ on $[1, 6]$ by using 5 slices.

3. Construct the general summation formula that specifically describes the approximation to the area bounded by the *x*- axis and the function $y = x \ln x$ between *x*=1 and *x*=*e*. Also, explain what each part of such formula represents geometrically.

4. Estimate the area of the region bounded by the function $y = e^2 - e^x$ in the first quadrant by using 4 inscribed rectangles. Clearly present the specific formula describing the approximation: pure calculator work will not give you many marks.

5. Compute a Riemann sum with *n*=6 and midpoint estimates to approximate the area of the region between the x-axis and the function $y = \cosh^2 x$ between *x*=1 and *x*=2.5.

6. Write an expression that provides an approximation to the area of the region bounded by $y = e^{-x^2}$, the *x*-axis, the *y*-axis and the line *x*=5. Also, explain the practical (geometric) meaning of each element of such expression.

In questions 7-8, identify a region whose area can be approximated by the given summation. Clearly explain the rationale you use to identify the key features of this region.

7. $\sum_{i=1}^{9} \cosh\left(2 + \frac{i}{3}\right)$

8. $\sum_{i=1}^{8} 3e^{4-\frac{i}{5}}$

Theory questions:

1. Identify three factors that contribute to making a Riemann sum an approximation of the real area in an applied problem.

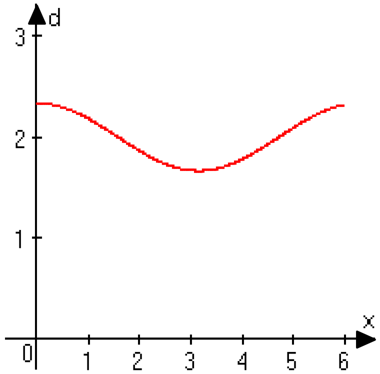
2. How can one improve the approximation given by a Riemann sum?

3. In the Riemann sum, what is the geometrical meaning of the expression inside the sigma notation?

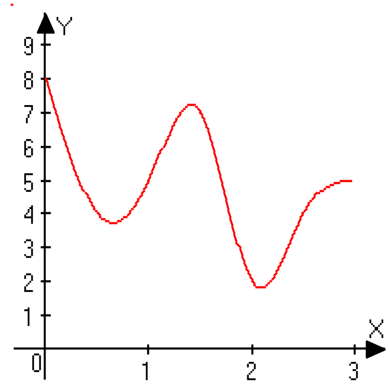
4. Is it necessary to have strips of equal width to construct a Riemann sum?

Application questions:

1. The linear density of a metal bar is represented by the function whose graph is given here. Explain how we can use approximating rectangles to estimate the area under this curve, use your method to obtain such an estimate, and determine what the physical meaning of such area is in relation to the bar.



2. A piece of property is bounded by three roads, represented by the *x* and *y* axes and the line $x = 3$, and a river whose course is represented by the curve below. Use proper sigma notation and 6 rectangular strips to estimate the area of such property.



Templated questions:

1. Write the correct sigma notation for any Riemann sum you encounter.

2. In each case where you used a Riemann sum to estimate an area, try to determine if you obtained an overestimate or an underestimate. However, keep in mind that this cannot always be done.

What questions do you have for your instructor?