The definite Riemann integral

What you need to know already:	*What you can learn here:*
➤ How to approximate the area under a curve by using Riemann sums.	➤ How to use the limit process to obtain the exact area under a curve.

We have seen that by using Riemann sums we can approximate the area under a positive and continuous function to any degree of accuracy we need. Here is a reminder of that process, when done under the simple assumption of equal width intervals.

Strategy to approximate the area under a function

Given a positive function $f(x)$, to obtain an approximate value for the area under it between two values *a* and *b*:

1. ***Slice*** the region into *n* thin vertical strips of width $\Delta x = \frac{b-a}{n}$.
2. ***Approximate*** the area of each strip by using rectangles of width Δx and height $f(x_i)$, where x_i is a suitably chosen value within the strip
3. ***Add up*** the areas of all such rectangles to obtain the Riemann sum

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

Although this is a very good practical solution to the problem, it is still unsatisfactory because:

- It is only an approximation.
- It only applies to areas under a positive function.
- It requires several choices, both for *n* and for each x_i, with each choice providing a different answer.
- It is difficult to implement, as it requires many function evaluations, with their number increasing if we want better approximations.
- The notation is fairly cumbersome.

But we have a calculus tool that will allow us to resolve the first issue and, hopefully, make progress on the others. That tool, of course, is limits, and here is how its inclusion completes our strategy.

Strategy to compute the area under a curve

Given a positive function $f(x)$, to compute the exact value for the area of the region under it between two values *a* and *b*:

1. ***Slice*** the interval $[a,b]$ into *n* thin vertical strips of width $\Delta x = \frac{b-a}{n}$.
2. ***Approximate*** the area of each strip by using rectangles of width Δx and height $f(x_i)$, where x_i is a value in the strip
3. ***Add up*** the areas of all such rectangles to obtain the Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x$
4. ***Compute the limit*** of the Riemann sum as the number of slices *n* approaches ∞. If this limit exists and does not depend on the choices involved, the area of the region is ***defined*** as:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

What do you mean by "if the limit exists?"

As we know, not all limits exist, so there are situations where the limit we seek may not be there. Or it may be that it is different according to how we choose the intervals and the values within each. So, we cannot start jumping up and down with excitement yet. Let me point out, once again, some of the remaining difficulties:

- We are defining the area of the region as the limit of the Riemann sum, but does this definition agree with what we usually mean by area? It turns out that it does in the usual cases, but not in some specialized situations that you may see later. For now we'll accept that it does
- Although this method may allow us to arrive at the exact area of the region, its idea is theoretical and it is not clear how easily it can be implemented in practice.
- The computational aspects of the procedure still look very complicated and most likely prohibitive to carry out.
- It is not clear when the value of the limit depends on the choice of *n*, of the type of Riemann sum used and on how fast we let *n* go to ∞.

Therefore, for now we need to take this idea with a grain of salt: it is a good idea, but its details and operational steps needs to be fleshed out still.

However, since I know (and probably you know as well) that these details will be resolved, let us work on the notation issue. This will, in fact, help us as we address the other issues.

Definition

If the $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ exists and does not depend on any of the choices made, then such limit is called the ***Riemann integral*** or ***definite integral*** of the function $f(x)$ between *a* and *b* and is denoted by:

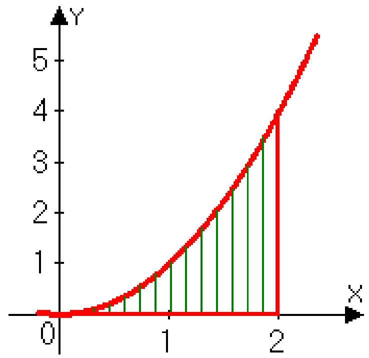
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

Example: $y = x^2, a = 0, \ b = 2$

The area under this curve is shaded in the picture, to remind you of the slicing done in the definition of its area. Its value is given by the Riemann integral:

$$\lim_{n\to\infty}\sum_{i=1}^{n} x_i^2 \Delta x = \int_0^2 x^2 \, dx$$

Notice how the notation on the right is leaner and clearer than the limit of a sum notation, and even just of the Riemann sum notation.



Why do we call this a "Riemann" integral or a "definite" integral? Why not just call it an integral?

Knots on your finger

A ***definite*** integral is a number, given by the limit specified in its definition, as opposed to an indefinite integral, which is a family of functions.

A ***Riemann*** integral is defined through the process we have seen here of slicing, approximating, adding and computing the limit. Other processes have been suggested that lead to definitions that are equivalent to the Riemann definition when the function is continuous, but not in other situations.

In the future I shall preferentially refer to these integrals as ***definite***, since we shall not see any other type of integral in this course besides the Riemann one. But do remember that there are other types.

And since we are at it, here are some further important considerations to keep in mind when dealing with definite integrals.

Knot on your finger

- The iconic symbol $\int$ that identifies a definite integral is indeed a long capital letter ***S*** and it stands for ***sum***, since it represents the limit of a sum: it is a *Sum* and the *limit* makes it long.
- As of now, there is ***no visible connection between*** the ***indefinite integral*** of a function, defined as the set of its antiderivatives, and this ***definite integral***, defined as a limit and representing an area. An important relation will be discovered later, but we are not there yet!
- Once we explore the connection between definite and indefinite integrals, ***it will become clear*** why indefinite integrals are also represented by the elongated *S* that is the integral sign.

There are three technical facts that I can now state for you, even though I will do so without proof. Such proofs require technical details beyond our goals and they are provided in many standard textbooks.

The first two are very theoretical and are here just for completeness and to eliminate their issue from our discussions.

Technical facts

- ***If the limit*** that defines a Riemann integral ***exists***, it is the ***same regardless*** of the choice of values we pick in each interval and regardless of how fast we let $n \to \infty$.
- In fact, it is ***not*** even ***necessary*** for the slices to have the ***same width***, as long as in the limit process the largest width goes to 0.

The last one, however, is very practical, as it allows us to have some guarantees that these integrals exist in relevant cases and motivates us to look for better ways to compute them.

Technical facts

- If $f(x)$ is ***continuous*** on $[a, b]$, then the integral $\int_a^b f(x)dx$ ***certainly exists***, although we may still have problems finding it.

Example: $\int_0^2 x^2\, dx$

This integral, which measures the area under the curve $y = x^2$ between $x = 0$ and $x = 2$ certainly exists, since the function we are using is continuous. However, computing it as the limit $\lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 \Delta x$ may still be very difficult.

Thank goodness, we shall soon see how to find it in an easy way!

On a less important, but interesting note, when Newton and Leibniz developed calculus, the idea of a definite integral was already around. It was Newton's teacher, Isaac Barrow, who studied this concept a lot and came very close to finding the key that opened up the treasures of calculus. What Newton and Leibniz did was identify the connection between definite and indefinite integrals, thus making the computation of the integral much easier. This unlocked the tremendous power that integrals now have in mathematics as well as all other sciences.

We are going to study the properties of these integrals in detail in later sections, so I will end this one by setting more terminology for such details. You are likely already familiar with some of them, but make sure to become *totally* familiar, since they will be used extensively.

Definition

In a definite integral $\int_a^b f(x)dx$:

- The function $f(x)$ is called the ***integrand***.
- The values ***a*** and ***b*** are called ***the limits of integration***.
- The symbol ***dx*** is called the ***differential*** of *x* and it is an important and necessary part of the integral, not to be omitted or ignored, since it corresponds to the width of the slices.

Example: $\int_0^2 x^2\,dx$

In this integral :

- The symbol $\int$ indicates the limit of a sum
- The function $f(x)=x^2$ is the integrand
- dx is the differential
- 0 and 2 are the limits of integration.

Make sure not to confuse the limits of integration, which are the boundaries of the region whose area we are computing and are very practical, with the limit needed to define the integral, which will soon disappear from our attention, replaced by a better method.

Remember that the differential was already an important part of an indefinite integral, as it identifies the variable of integration and plays an important role in any changes of variable. Now we see that it also has a geometric meaning: it represents the width of each slice. So, don't be sloppy when dealing with it: it deserves your respect and attention. Looking at the connection between the two types of integrals will be our next step, after a short exploration of the basic properties of the Riemann integral.

Summary

- The limit of a Riemann sum, as $n \to \infty$ can be used to determine the area under a curve.
- The more efficient notation $\int_a^b f(x)\,dx$ is used for such limit and it is called the definite or Riemann integral of the function between the two limits of integration.
- If that limit exists, it is the same, regardless of any choices involved.
- When the integrand is continuous over the interval defined by the limits of integration, then the definite integral exists.

Common errors to avoid

- Become familiar with the notation and terminology and use both correctly on a consistent basis.

Learning questions for Section I 4-4

Review questions:

1. Explain what a Riemann integral is and how it is connected to the area problem.

2. Identify the notation and terminology used for each part of a definite integral.

Memory questions:

1. What is the notation for the definite integral of $y = f(x)$ on $[a,b]$?

2. Which simple condition on $y = f(x)$ guarantees that its definite integral over $[a,b]$ exists?

3. What is added to a Riemann sum to make it into a Riemann integral?

4. What is the geometrical meaning of the dx in the notation of a definite integral?

Computation questions:

For each of the integrals presented in questions 1-8:

a) use your calculator to sketch the region whose area is represented by the given integral
b) construct the limit formula that defines such integral
c) describe the geometric meaning of each part of that formula

1. $\int_0^2 \left(2x - x^2\right) dx$

2. $\int_0^{\pi} \tan \frac{x}{4} dx$

3. $\int_{\pi}^{2\pi} \left(3 + \sin x - \cos x\right) dx$

4. $\int_{1/2}^{1} x \ln x dx$

5. $\int_1^4 \left(3x - \frac{1}{\sqrt{x}}\right) dx$

6. $\int_{-1}^{0} x e^x dx$

7. $\displaystyle\int_1^2 \frac{1+\sin x^2}{x}\,dx$

8. $\displaystyle\int_{-2}^{-1} \frac{\cos x^2}{x}\,dx$

Express each of the limits presented in questions 9-12 as a definite integral.

9. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{ie^{i/n}}{n^2}$

10. $\displaystyle\lim_{n\to\infty}\frac{\pi-e}{n}\sum_{i=1}^{n}\sin\ln x_i$.

11. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(5+3\frac{i}{n}\right)^7$

12. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3+\sin\frac{i\pi}{n}-\cos\frac{i\pi}{n}}{n}$

13. Determine the value of the integral $\displaystyle\int_{-3}^{3}\left(4-\sqrt{9-x^2}\right)dx$ by identifying it as the area of a simple region and using appropriate geometric formulae.

14. Determine the value of the integral $\displaystyle\int_{-3}^{3}\left(\sqrt{9-x^2}-4\right)dx$ by identifying it as the area of a simple region and using appropriate geometric formulae.

15. Use a left point estimate to approximate the value of $\displaystyle\int_0^3 xe^{-x}\,dx$ with n=6.

Theory questions:

1. When does a definite integral represent an area?

2. Why do we use the symbol $\int$ to represent integrals?

3. In a Riemann integral, is it necessary that all rectangular strips have the same width?

4. The area of which familiar geometrical shape is represented by an integral of the form $\int_0^8 c^2 x\,dx$?

5. What single geometrical shape corresponds to the region whose area is given by $\int_0^1 (3-x)\,dx$?

6. In order to define a Riemann integral, is it necessary for the integrand to be positive?

7. The area under the curve $y = x^2$ between $x = 0$ and $x = 2$ is given by the integral $\int_0^2 x^2\,dx$. Here *x* and *dx* are measured in units of length and the integrand is a quadratic function. So why is the value of the integral an area, measured in square units, instead of the cubic units (volume) that are suggested by the product $x^2 dx$?

8. Why can't we assume that $\int_{-1}^{1} \frac{\cosh x}{x}\,dx$ exists?

What questions do you have for your instructor?