Chapter 4: Definite integrals and the FTC **Section 5**

General properties of definite integrals

What you need to know already:	What you can learn here:
➤ What a definite Riemann integral is.	➤ Some key properties of this integral that will allow us to use it effectively later.

We now have a formal definition of definite integral that theoretically allows us to compute areas under a curve for continuous functions. But only theoretically! That definition does not tell us how to compute a definite integral, unless we want to wonder through some complicated, possibly mind-boggling limit computations. Still, in the spirit of mathematical inquiry, let us see what properties we can extract from this definition: maybe some of these properties will tell us how to compute it efficiently.

In what follows, let us assume that $f(x)$ is continuous on $[a,b]$. We start from an obvious, but useful little fact.

Technical fact

If $f(a)$ exists $\int_a^a f(x)dx = 0$.

Proof

In this case there is no area, since we are talking about a line segment.

Another way to see this is that the base of any theoretical interval we may consider is 0.

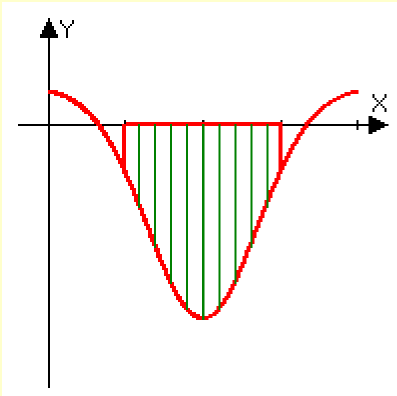
Next, notice that the definite integral is defined through an algebraic formula and therefore the assumption that the integrand be positive is not really needed. However, we do need to be careful with the interpretation of what the integral of a negative function represents.

Knot on your finger

If $f(x) \leq 0$ on $[a,b]$, then we can still define:

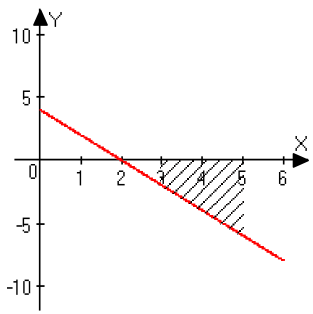
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

However, this integral, if it exists, is negative, being the limit of a sum of negative quantities. It therefore represents the ***negative of the area*** above the curve, as represented in this graph:



$$A = -\int_a^b f(x)\,dx$$

Example: $\int_3^5 (4-2x)\,dx$



We can compute the value of this integral by using appropriate geometric formulae! Notice that this integral represents the area of the region shaded in this graph.

But this is a trapezoid below the *x*-axis, hence the integral will equal the negative of its area. Therefore:

$$\int_3^5 (4-2x)\,dx = -\frac{(B+b)h}{2} = -\frac{\left[(4-2\times 5)+(4-2\times 3)\right]\times 2}{2} = -8$$

How do we deal with functions that are sometimes positive and sometime negative? To figure that out, we notice the following simple fact.

Technical fact
The additive property of definite integrals for their limits

If $y = f(x)$ is a function that is continuous on $[a,b]$, and $a < c < b$, then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof

Well, this is only an intuitive sketch of the proof, but the details are not difficult to check.

If $f(x) \geq 0$ on $[a,b]$, the equation simply states that the area under the curve from *a* to *b* can be considered as the sum of two areas under the curve, one from *a* to *c*, the other from *c* to *b*.

In the more general case, we use the algebraic definition of the definite integral as the limit of a sum and notice that the above split is simply reflecting the split of the sum into two sums, each containing a portion of the terms. In both cases the conclusion is correct.

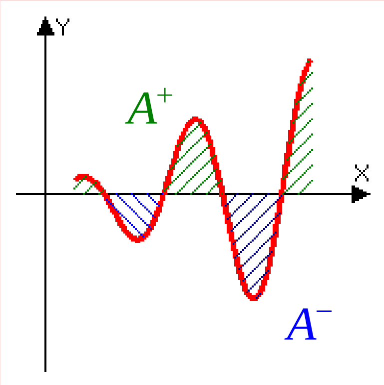
This property allows us to understand the meaning of any definite integral in terms of areas.

Technical fact

If $y = f(x)$ is any function that is ***continuous*** on $[a,b]$, we can still define its definite integral as:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

If the integral exists, it represents the difference between the area of the region bounded by the curve above the *x* axis and the area of the region bounded by the curve below the *x* axis:



$$\int_a^b f(x)dx = A^+ - A^-$$

The next properties are also fairly intuitive and will prove very useful when working more intensely with definite integrals.

Technical fact

If $y = f(x)$ is a function that is continuous on $[a,b]$, then:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Proof

This fact is justified by noticing that by going in the reverse direction along the *x* axis, we are using negative widths for the rectangular slices, so that the differentials have different signs on the two sides and therefore provide opposite values for the integrals.

Technical fact

If $f(x)$ and $g(x)$ are continuous functions on $[a,b]$, then:

$$\int_a^b \left(f(x)+g(x)\right)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Proof

Here we just need to remember that an integral is a limit of a sum and that sums create no problems when computing limits. Therefore the two terms of the sum can be separated into to different integrals. In formula:

$$\int_a^b \left(f(x)+g(x)\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(f(x_i)+g(x_i)\right)\Delta x$$
$$=\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x+\sum_{i=1}^{n} g(x_i)\Delta x\right)$$
$$=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x+\lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x$$
$$=\int_a^b f(x)dx+\int_a^b g(x)dx$$

There is another property that is very similar to the last one.

Technical fact

If $f(x)$ is continuous on $[a,b]$ and k is any real number, then:

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

I guess that the proof here is very similar to the previous one.

Certainly, and you will find the challenge of developing in the learning activities.

The last two facts are reminiscent of the corresponding properties for indefinite integral. However, remember that so far, antiderivatives are not part of anything we have done!

We are now ready to move to the real meat and potato of this chapter, to the reason why calculus is such a big deal in the history of science and technology. So, on to the next two sections!

What, no examples?

There will be plenty in those sections. For now, just reflect on the meaning of the facts we have seen in this section and check that you understand such meanings through the learning questions. Remember that we still don't know how to compute any of these integrals efficiently and – believe me – you don't want to compute any of them by using the limit definition, just as you did not like to do compute derivatives through the definition!

OK, I look forward to what is coming then.

Summary

- A definite integral may be defined for any function, not just positive ones, but its interpretation in terms of areas must be modified.
- In general, the definite integral of a function represents the difference between the areas of the regions bounded by the function above the x axis and those bounded below it.
- Definite integrals can be split along the limits, or along a sum of the integrand. Both properties end up being useful in applications.

Common errors to avoid

- Just because the theoretical facts listed in this section seem reasonable, it does not mean that they are clear to you! Challenge yourself with as many questions about them as possible and make sure you can explain your arguments and conclusions to anyone, eh?

Learning questions for Section I 4-5

Review questions:

1. Explain the relationship between the definite integral of a continuous function and the area of the regions such function bounds.
2. Provide a clear explanation of why definite integrals have the addition properties with respect to the integrand and to the limits of integration.

Memory questions:

1. When does a definite integral represent an area?
2. State the addition property of the definite integral with respect to the limits.
3. State the addition property of the definite integral with respect to the integrand.
4. What happens to a definite integral if we switch the limits of integration?

Computation questions:

In questions 1-4, use a Riemann integral notation to express the exact value of the area of the region bounded by the given curves.

1. $y = x^2 - 6x + 8,\ y = 0,\ x = 4,\ x = 6$
2. $y = x^3 - 6x + 8,\ y = 0,\ x = -2,\ x = 2$
3. $y = \cos x,\ y = 0,\ x = 0,\ x = \pi$
4. $y = \cos x,\ y = 0,\ x = -\frac{\pi}{2},\ x = \frac{\pi}{2}$

In questions 5-10, express the given integral in terms of areas of regions.

5. $\int_{-2}^{2}\left(4-x^2\right)dx$

6. $\int_{-1}^{1}\left(4-x^2\right)dx$

7. $\int_{-4}^{4}\left(4-x^2\right)dx$

8. $\int_{2}^{4}\left(4-x^2\right)dx$

9. $\int_{1}^{5}\ln x dx$

10. $\int_{0.5}^{5}\ln x dx$

In questions 11-12 use basic geometric formulae to determine the value of the given integral.

11. $\int_{-1}^{1}\sqrt{1-x^2}dx$.

12. $\int_{0}^{3}\left(2+\sqrt{9-x^2}\right)dx$

13. If $f(x)$ is a continuous function for which $\int_{5}^{8} f(x)dx = 9$ and $\int_{2}^{8} f(x)dx = 5$, determine $\int_{2}^{5} f(x)dx$ and clearly explain how you obtain your conclusion.

14. Use properties of definite integrals to determine the value of $\int_{1}^{-1}\left(x^3-x\right)dx$.

Theory questions:

1. When you are asked to compute a definite integral with constant limits, should your conclusion consist of a constant, a function, a family of functions or something else?

2. Present a definite integral having $y = x$ as integrand and having 0 as value.

3. How is the value of $\int_{2\pi}^{\pi}(3+\sin x-\cos x)dx$ related to the value of $\int_{\pi}^{2\pi}(3+\sin x-\cos x)dx$?

4. What is the geometric interpretation of the integral $\int_1^{10} \cos x dx$?

5. What is the geometrical interpretation of the differential dx in a definite integral?

6. In what situations is a definite integral negative?

7. Does the integral $\int_0^1 \left(x^2 - 4\right) dx$ represent the area of a region?

8. The area under the curve $y = x^2$ between 0 and 1 is given by the integral $\int_0^1 x^2 dx$. Here x and dx are measured in units of length and the integrand is a square function. So, why is the value of the integral an area, measured in square units instead of a volume, measured in cubic units?

9. What is a simple procedure to use when computing the area of a region described with x as a function of y?

Proof questions:

1. Prove that $\int_a^b k dx = k\left(b - a\right)$ for any constants a, b and k.

2. Prove that if $f\left(x\right)$ is continuous on $\left[a, b\right]$, then $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

3. Show that if a and b are two fixed points of the continuous and invertible function $y = f\left(x\right)$ and $b > a > 0$, then:

$$\int_a^b \left(f\left(x\right) + f^{-1}\left(x\right)\right) dx = b^2 - a^2 .$$

(Courtesy of Bill Ifemeje, former student)

What questions do you have for your instructor?