Chapter 4: Definite integrals and the FTC — Section 8

# Change of variables and the FTC

| What you need to know already: | What you can learn here: |
|---|---|
| ➤ The practical version of the FTC and the method of change of variables for integration. | ➤ How to combine the two in an effective way. |

The FTC, in its practical version, tells us that to compute a definite integral we need to compute an antiderivative of the integrand and evaluate it at the two limits of integration:

$$\int_a^b f(x)dx = F(b) - F(a)$$

In many situations, in order to compute the needed antiderivative we may have to use the method of change of variables. What happens to the limits of integration when we do that?

## Warning bells

The option of ***leaving*** the limits of integration as they are when using the method of substitution is ***NOT CORRECT***, since those limits refer to values of the original variable, NOT of the new variable that has been introduced.

When you think about it, this seems to be an obvious fact and the warning unnecessary. Yet continuing to use the original limits with the new variable, thus ending up with an incorrect conclusion, is a common beginner's mistake. This is such a common, major and avoidable error that I decided to highlight it by giving it its own section!

And, to help you avoid it, besides the above warning, here are three correct ways to proceed.

## Strategies for Changing variables in a definite integral

When applying the FTC to a definite integral that requires the method of ***substitution***, then the limits of integration can be handled in one of these ways:

- Compute the needed ***antiderivative separately***, as an indefinite integral, and then use it in the FTC formula.
- Implement the ***change*** of variables to ***the entire definite integral***, ***including the limits*** of integration, and then apply the FTC to the new integral so obtained.

***Example:*** $\int_0^1 x^4\sqrt{2x^5+5}\,dx$

To determine the value of this integral, we use the FTC. Computing the antiderivative requires the substitution $u = 2x^5+5,\ du = 10x^4dx$. We can therefore proceed as follows.

***Option 1***: Compute the antiderivative separately:

$$\int x^4\sqrt{2x^5+5} = \frac{1}{10}\int\sqrt{u}\,du = \frac{1}{10}\frac{2u^{3/2}}{3} = \frac{(2x^5+5)^{3/2}}{15}+c$$

We can now apply the FTC by using the simplest antiderivative:

$$\int_0^1 x^4\sqrt{2x^5+5}\,dx = \left[\frac{(2x^5+5)^{3/2}}{15}\right]_0^1 = \frac{7^{3/2}}{15}-\frac{5^{3/2}}{15}$$

***Option 2***: Apply the substitution to the limits as well, noticing that:

$$u = 2x^5+5 \quad\Rightarrow\quad \begin{cases} x=0 \quad\Rightarrow\quad u=5 \\ x=1 \quad\Rightarrow\quad u=7 \end{cases}$$

In this way the original integral becomes

$$\int_0^1 x^4\sqrt{2x^5+5}\,dx = \frac{1}{10}\int_5^7\sqrt{u}\,du$$

$$= \frac{1}{10}\left[\frac{2u^{3/2}}{3}\right]_5^7 = \frac{1}{10}\left[\frac{2}{3}7^{3/2}-\frac{2}{3}5^{3/2}\right]$$

The two conclusions are equal, as you can check with a little factoring.

### *You said that there were three methods, but you only listed two.*

Very observant! That is because the third method is correct, but cumbersome, so I do not recommend it. Still, you may want to try and see if you like it.

## ***Knot on your finger***

When using the method of substitution on a definite integral, one option is to ***keep the original limits***, but clearly ***indicate them as such***, and then use them only after returning to the original variable.

***Example:*** $\int_0^1 x^4\sqrt{2x^5+5}\,dx$

With this approach, we still use the substitution $u = 2x^5+5,\ du = 10x^4dx$, but write the steps as follows.

$$\int_0^1 x^4\sqrt{2x^5+5}\,dt = \frac{1}{10}\int_{x=0}^{x=1}\sqrt{u}\,du = \frac{1}{10}\left[\frac{2u^{3/2}}{3}\right]_{x=0}^{x=1}$$

$$= \left[\frac{(2x^5+5)^{3/2}}{15}\right]_0^1 = \frac{7^{3/2}}{15}-\frac{5^{3/2}}{15}$$

Same conclusion, of course, but more writing work with limits and variables.

### *So, which method should we use?*

Whichever you prefer. Just do not mix them up, thus ending up with the integrand in terms of one variable and the limits in terms of another.

I want to finish this section by showing you how to compute a very tricky definite integral by using substitutions in a very clever and non-intuitive way. Be prepared for some mathematical fireworks! And no, it will not be in any exam!

**Example:** $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}dx$

If you try to compute this integral by finding an antiderivative for the integrand, good luck! But some clever substitutions (and trigonometry) does lead to the answer!

Notice that the denominator is of the form for which we used for the inverse trig substitution $x=\tan\theta,\ dx=\sec^2\theta d\theta$, so we try it:

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx=\int_0^{\pi/4}\frac{\ln(\tan\theta+1)}{\tan^2\theta+1}\sec^2\theta d\theta=\int_0^{\pi/4}\ln(\tan\theta+1)d\theta$$

Now we rewrite the argument of the logarithm by using its properties:

$$=\int_0^{\pi/4}\ln\left(\frac{\sin\theta+\cos\theta}{\cos\theta}\right)d\theta=\int_0^{\pi/4}\left[\ln(\sin\theta+\cos\theta)-\ln(\cos\theta)\right]d\theta$$

I will leave to you, as a refresher trig exercise, to use the addition formula for cosine to prove that $\sin\theta+\cos\theta=\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)$. Since that is true, our integral becomes:

$$=\int_0^{\pi/4}\ln\left(\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right)d\theta-\int_0^{\pi/4}\ln(\cos\theta)d\theta$$

$$=\int_0^{\pi/4}\ln\sqrt{2}d\theta+\int_0^{\pi/4}\ln\left(\cos\left(\theta-\frac{\pi}{4}\right)\right)d\theta-\int_0^{\pi/4}\ln(\cos\theta)d\theta$$

The first integral is easy, while on the second we apply the substitution $u=\frac{\pi}{4}-\theta,\ du=-d\theta$:

$$=\frac{\pi\ln\sqrt{2}}{4}-\int_{\pi/4}^{0}\ln(\cos(-u))du-\int_0^{\pi/4}\ln(\cos\theta)d\theta$$

Since the cosine function is even and by reversing the limits, we have:

$$=\frac{\pi\ln\sqrt{2}}{4}+\int_0^{\pi/4}\ln(\cos u)du-\int_0^{\pi/4}\ln(\cos\theta)d\theta$$

But the last two integrals are the same, except for a different name of the variable, and they have opposite signs, so they cancel and we conclude that:

$$\int_0^1\frac{\ln(x+1)}{x^2+1}dx=\frac{\pi\ln\sqrt{2}}{4}$$

*Wicked! Such trickery!*

Yes, but it works! Use this example to get an appreciation of the steps available through change of variables, of the meaning of definite integrals and of some unfamiliar trig relationships. But again, don't expect this level of trickery on any test!

# Summary

- When using the method of substitution on a definite integral, make sure to deal with the limits of integration correctly, by always keeping them consistent with the variable of integration used.

# Common errors to avoid

- Don't make the common error of mixing up variable and limits: it does not look good on your tests or your resume!

# Learning questions for Section I 4-8

## Review questions:

1. Describe the two options available for dealing with the integration limits when using the method of substitution on a definite integral.

## Computation questions:

Determine the exact value of the definite integrals presented in questions 1-13.

1. $\displaystyle\int_0^1 \frac{x^3}{\left(x^2-4\right)^3}\,dx$ .

2. $\displaystyle\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx$ .

3. $\displaystyle\int_{\sqrt{\pi/6}}^{\sqrt{\pi/4}} x\tan x^2 dx$

4. $\displaystyle\int_0^1 \cos\sqrt{x}\,dx$

5. $\displaystyle\int_0^{1/\sqrt{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx$

6. $\displaystyle\int_0^1 \frac{x^2}{(4+x^2)^2}\,dx$

7. $\int_0^1 e^{\sqrt{x}}\,dx$

8. $\int_{\pi/4}^{\pi/2} \tan^2 x \sec^4 x\,dx$

9. $\int_2^4 \frac{\sin(\pi/x)}{3x^2}\,dx$

10. $\int_0^1 \frac{2}{3+\sqrt[4]{x}}\,dx$

11. $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

12. $\int_0^{16} \frac{3}{2+\sqrt[4]{x}}\,dx$

Use a suitable substitution and the method of partial fractions to compute the values of the following integrals:

13. $\int_0^1 \frac{2}{3+\sqrt[4]{x}}\,dx$

14. $\int_0^1 \frac{2}{x^{1/4}+2x^{3/4}}\,dx$

## *Theory questions:*

1. What are the two basic ways to apply the method of change of variables to a definite integral?

## *Proof questions:*

1. Assume that $y = f(x)$ is a continuous function whose values are always positive on the interval $[a,b]$.

   a) Prove that $\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx = \frac{b-a}{2}$. Hint: use the only reasonable substitution and remember that in a substitution, the new variable is fictitious.

   b) Use the above result to compute $\int_2^4 \frac{\sqrt{x}}{\sqrt{6-x}+\sqrt{x}}\,dx$ and $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$.

2. Prove that if $a$ and $b$ are positive real numbers and $f(x)$ is integrable on $[-a, a]$, then $\int_{-a}^{a} \frac{f(x)+1}{b^x+1}\,dx = a + \int_0^a f(x)\,dx$. This proof relies on a clever method that is discussed in this video.

3. A student performs the following integration:

$$\int_4^9 \frac{dx}{x-\sqrt{x}} = \left[\ln\left|x-\sqrt{x}\right|\right]_4^9 = \ln 6 - \ln 2 = \ln 4$$

Identify all reasons why this is incorrect and show, by using proper methods, that the conclusion is still right!

## *What questions do you have for your instructor?*