Chapter 9: Orthogonality | *Section 2*

Orthogonal matrices

What you need to know already:	*What you can learn here:*
➤ What orthogonal and orthonormal bases for subspaces are.	➤ A special kind of matrices that are not pretty, but are nice!

We have seen that orthogonal and orthonormal bases have some nice properties that make them worth noticing. In this section you'll see that those nice properties transfer to, and get exploited by a nice type of matrices. That is, in this section we'll see how the property of being orthogonal – a concept that was born in geometry – becomes a nice property for matrices, which are algebraic objects. This interplay between algebra and geometry is another exciting aspect of linear algebra that we shall explore a little further in the next chapter. But let us not get astray. Here is the key definition.

Definition

An $n \times n$ matrix $\mathbf{Q}$ is said to be ***orthogonal*** if its columns form an ***orthonormal basis*** for $\mathbb{R}^n$, that is, if the dot product of

- any column with itself is 1, and
- any two different columns is 0.

Before I say anything else, let me point out the obvious discrepancy in this definition: we call this matrix ***orthogonal*** if its columns form an ***orthonormal*** basis, NOT just an orthogonal one. If it were me, I would have called these matrices *orthonormal*, but the first mathematicians who defined and used them probably considered this word too long and settled for the shorter word, trusting that other scholars would get and appreciate the distinction. Alas, many beginner students don't! So, keep this jargon issue in mind and make an effort to enter the club of scholars who get it!

Fair warning, but what is so good about such matrices, besides the fact that they identify an orthonormal basis?

In fact, we shall see soon that they identify *two* orthonormal bases, but that is not the main point. As I said, they are nice matrices in their own right. Here is one of them for $\mathbb{R}^3$:

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & -\frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \end{bmatrix}$$

Yuck! And you call this nice?

Sure: would you like to compute its inverse?

That must be a horrible exercise and I don't even know if it is invertible!

If you let go of your revulsion for the strange numbers that make up the matrix and trust your brains instead, you will quickly realize the following two important facts about orthogonal matrices.

Technical fact

Any ***orthogonal*** matrix $\mathbf{Q}$ is ***invertible***.

Proof

Since the columns of $\mathbf{Q}$ form a basis (never mind that it is a special basis), they are linearly independent. As we have seen before, that is enough to tell us that the determinant is not 0 and hence that the matrix is invertible.

Oh, right!

But there is more…

Technical fact

If $\mathbf{Q}$ is an ***orthogonal*** matrix, ***then*** $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

If $\mathbf{Q}^{-1} = \mathbf{Q}^T$, ***then*** $\mathbf{Q}$ is ***orthogonal***.

What?! That easy?

Yes, and simple to prove.

Proof

If we consider the product $\mathbf{Q}^T\mathbf{Q}$, we notice that each one of its entries is the dot product of two columns of $\mathbf{Q}$. But such product is 1 if we multiply a column by itself, that is, if we are on the diagonal of the product, and 0 elsewhere. But that tells us that the product is the identity matrix and, therefore, that $\mathbf{Q}^T$ must be the inverse of $\mathbf{Q}$.

The same argument, in reverse, proves the second claim.

Wow, that is neat indeed!

I thought so. Therefore, if I ask you to compute the inverse of an orthogonal matrix, all you have to do is transpose it!

So, the inverse of $\mathbf{Q} = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & -\frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \end{bmatrix}$ *is* $\mathbf{Q}^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{2}{3\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{-5}{3\sqrt{5}} \end{bmatrix}$*!*

Yes, but there are more goodies to come: that is why I said that orthogonal matrices are not pretty, but are nice!

Technical fact

If $\mathbf{Q}$ is an orthogonal matrix, then so are both $\mathbf{Q}^{-1}$ and $\mathbf{Q}^T$

Of course: they are the same matrix and $\left(\mathbf{Q}^{-1}\right)^{-1} = \left(\mathbf{Q}^{T}\right)^{-1} = \left(\mathbf{Q}^{T}\right)^{T} = \mathbf{Q}$.

Yes. More good stuff to come.

Technical fact

The determinant of an orthogonal matrix can only be 1 or -1.

Wait, let me think this one through by myself.

Good idea: proof sent to the *Learning questions*. Let's see more.

Technical fact

If **Q** is an ***orthogonal*** matrix, then it preserves the dot product, in the sense that for any two vectors **v** and **w** of the proper dimension:

$$\mathbf{Qv}\bullet\mathbf{Qw} = \mathbf{v}\bullet\mathbf{w}$$

Proof

All we need is basic properties of matrix algebra:

$$\mathbf{Qv}\bullet\mathbf{Qw} = (\mathbf{Qv})^T\,\mathbf{Qw} = \mathbf{v}^T\mathbf{Q}^T\mathbf{Qw} = \mathbf{v}^T\mathbf{Iw} = \mathbf{v}\bullet\mathbf{w}$$

The proof of the next one follows naturally, so I will leave it too to the *Learning questions*.

Technical fact

If **Q** is an orthogonal matrix, then it preserves the norm, in the sense that for any vector **v** of the proper dimension:

$$\|\mathbf{Qv}\| = \|\mathbf{v}\|$$

The last two facts may be interpreted by saying that orthogonal matrices are very gentle when multiplied on the left of other vectors. This will come in handy in what we shall learn in the next chapter.

I am starting to see why orthogonal matrices are so nice! What else is there?

Oh, we could say a lot more, but it is so much fun that I will leave some of the easiest discoveries for the *Learning question*!

By the way, did you notice that this is a very theoretical section? There are applications of these properties, but guess what!

We need to learn more things before we can see them?

Yep! One of them is in the *Learning questions*, though: have fun!

Wait, one more question: why do we use $\mathbf{Q}$ *to denote an orthogonal matrix?*

No technical reason, really. Just a tradition. Its staying power is probably due to some of those later facts, some of which have acquired a name involving $\mathbf{Q}$, such as the $\mathbf{QR}$ decomposition of a matrix that you will see in a later section. If I find out more, I will let you know!

Summary

- An orthogonal matrix is a square one whose columns form an orthonormal basis.
- Orthogonal matrices preserve dot product and norm and have many more nice properties.

Common errors to avoid

- Clarify the fact that an orthogonal matrix consists of an orthonormal (not just orthogonal) basis.
- Don't be scared by the strange values that are the entries of an orthogonal matrix: its beauty is in its properties.

Learning questions for Section LA 9-2

Review questions:

1. Describe what makes a matrix orthogonal.
2. Identify two properties that make orthogonal matrices "*nice*".

Memory questions:

1. What property defines an orthogonal matrix?
2. What is the easiest way to check if a matrix is orthogonal?
3. What values can the determinant of an orthogonal matrix take?

Computation questions:

1. Check that the matrix $\begin{bmatrix} \sqrt{3}/4 & -\sqrt{3}/2 & -1/4 \\ 3/4 & 1/2 & -\sqrt{3}/4 \\ 1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$ is orthogonal.
2. Provide a simple reason explaining why the matrix $\begin{bmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ is not orthogonal.

3. Determine whether the matrix $\begin{bmatrix} 1 & 0 & 0 & 1/\sqrt{6} \\ 0 & 2/3 & 1/\sqrt{2} & -2/\sqrt{6} \\ 0 & -2/3 & 1/\sqrt{2} & 3/\sqrt{6} \\ 0 & 1/3 & 0 & 2/\sqrt{6} \end{bmatrix}$ is orthogonal and provide at least two reasons for your conclusions.

4. Check whether the matrix $\mathbf{A} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is orthogonal and then compute the length of the vector $\mathbf{Ax}$, where $\mathbf{x}^T = \begin{bmatrix} \sqrt{3} & \sqrt{5} & \sqrt{8} \end{bmatrix}$.

5. Determine all the vectors that can go in the last column of the matrix $\begin{bmatrix} .5 & -.5 & -.5 & w \\ .5 & -.5 & .5 & x \\ .5 & .5 & -.5 & y \\ .5 & .5 & .5 & z \end{bmatrix}$ in order to make it an orthogonal matrix.

6. Determine all the possible values of x and y, if any, for which $\begin{bmatrix} x & 3y & x & 3y \\ x & -5y & y & y \\ 3y & x/3 & y & -5y \\ x & y & -5y & y \end{bmatrix}$ is an orthogonal matrix.

7. Given the matrix $\mathbf{A} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$, use the simplest possible method to determine:
   a) if it is orthogonal
   b) its inverse
   c) the length of the vector $\mathbf{A}\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$

Theory questions:

1. If **Q** is an $n \times n$ orthogonal matrix, what do its rows form for $\mathbb{R}^n$?
2. What is the simplest way to obtain the inverse of an orthogonal matrix?
3. If **Q** is an orthogonal matrix, what matrix is $\mathbf{Q}^T\mathbf{Q}$?
4. If the determinant of a 6×6 matrix is -1, can the matrix be orthogonal?
5. If **Q** is an orthogonal matrix, what is the determinant of $\mathbf{Q}^4$?
6. If **A** is orthogonal, how many solutions does the system **AX**=**0** have?
7. If a matrix is orthogonal, do its rows form an orthonormal basis?
8. Is the product of two orthogonal matrices orthogonal?
9. Is the sum of two orthogonal matrices orthogonal?
10. If **Q** is an $n \times n$ orthogonal matrix, are $\mathbf{Q}^T$ and $\mathbf{Q}^{-1}$ also orthogonal?

Proof questions:

1. Prove that the product of two orthogonal matrices is still orthogonal.

2. Prove that if $\mathbf{Q}$ is an orthogonal matrix, then $\|\mathbf{Qv}\| = \|\mathbf{v}\|$ for any vector $\mathbf{v}$ of suitable dimension.

3. Prove that if $\mathbf{Q}$ is an orthogonal matrix, then $\mathbf{Qv} \bullet \mathbf{Qw} = \mathbf{v} \bullet \mathbf{w}$ for any vectors $\mathbf{v}$ and $\mathbf{w}$ of suitable dimension.

4. Prove that if $\mathbf{Q}$ is an orthogonal matrix, then the angle between $\mathbf{Qv}$ and $\mathbf{Qw}$ is the same as the angle between $\mathbf{v}$ and $\mathbf{w}$ for any two vectors $\mathbf{v}$ and $\mathbf{w}$ of suitable dimension.

5. Prove that the determinant of an orthogonal matrix can only be 1 or -1.

6. Show that any orthogonal 2×2 matrix is of the form $\begin{bmatrix} \cos\theta & \pm\sin\theta \\ \pm\sin\theta & \pm\cos\theta \end{bmatrix}$ for some suitable choice of sign. Also, determine which combinations of signs provide an orthogonal matrix and which ones don't.

Templated questions:

1. Pick a 3×3 orthogonal matrix (you may use any one presented in this section) and two 3D vectors and verify that all properties related to their dot product and angle that have been presented in this section (including the *Learning questions*) are true.

What questions do you have for your instructor?