Chapter 4: Special types of functions | **Section 1**

Two variables functions

What you need to know already:	What you can learn here:
➤ What a function is and what their domain and range are.	➤ How to extend the concept of function to allow for two (or more) input variables.

All functions we have seen so far are of the basic variety: processes that change a single input, usually denoted by x, to a single output, usually denoted by y, thus dealing with relations of the form $y = f(x)$. But this is a very simple approach that does not reflect the much bigger complexity we find in real world phenomena.

In this section and the next we shall take two very small steps towards considering relations that are a little closer to reality.

Definition

A ***two-variable function***, also called a ***function of two variables***, is a procedure, usually denoted by f, that associates to a pair of real numbers (x, y) a unique number, usually denoted by $z = f(x, y)$ and called the ***image*** of (x, y).

So, evaluating a two-variable function is done as for usual functions, eh?

Correct. And so are all other basic operations with functions.

Example: $f(x,y) = \sqrt{x-y}$; $g(x,y) = x^2 + \sin y$

These are two-variable functions such that, for instance:

$$f(5,1) = \sqrt{5-1} = 2 \quad ; \quad g(1,\pi) = 1^2 + \sin\pi = 1$$

$$f(-5,-14) = \sqrt{-5+14} = 3 \quad ; \quad g\left(3,\frac{\pi}{2}\right) = 3^2 + \sin\frac{\pi}{2} = 10$$

$$f(7,-5) = \sqrt{7+5} = \sqrt{12} \quad ; \quad g(-2,0) = (-2)^2 + \sin 0 = 4$$

We can also combine these two functions algebraically:

$$f + g = \sqrt{x-y} + x^2 + \sin y$$

$$f \times g = \sqrt{x-y}\left(x^2 + \sin y\right)$$

$$f\left(g(x,y),y\right) = \sqrt{x^2 + \sin y - y}$$

$$g\left(x, f(x,y)\right) = x^2 + \sin\sqrt{x-y}$$

Because of the presence of two input variables, more combinations are possible and some may present additional problems, but we shall look at those later.

Other concepts related to regular functions can be extended to two-variable functions, with the most relevant, for now, being the following.

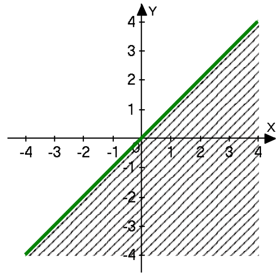
Definition

The ***domain*** of a two variable function is the set of pairs (x, y) for which it can be computed.

The ***range*** of a two variable function is the set of images it produces.

Example: $z = \sqrt{x - y}$

The domain of this function consists of all pairs (x, y) for which $x \geq y$. This corresponds to the region in the x-y plane that is on or below the line $y = x$, as shown in this picture.



The range of this function includes all non-negative numbers, since we can get $f(x, y) = 0$ by choosing $x = y$, we can get any positive number a by using $(a^2, 0)$, but we cannot get any negative numbers, since the square root never produces negative values.

Example: $z = x^2 + \sin y$

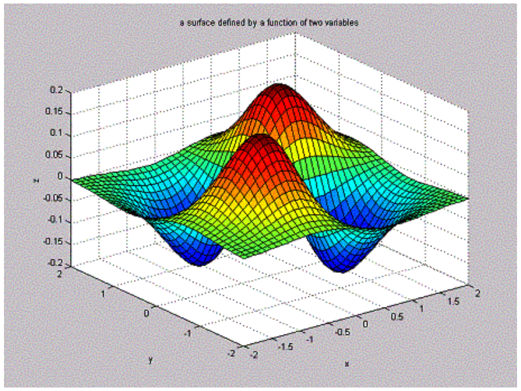
The domain of this function consists of all pairs (x, y), since neither the square nor the sine function have any special requirements. Since $x^2 \geq 0$ and $-1 \leq \sin y \leq 1$, the range of this function includes all numbers not lower than -1.

Knots on your finger

While the domain of a usual function consists of a set of numbers, represented by sets on the real line, the domain of a two-variable function consists of ***pairs of numbers***, represented by ***regions*** in the *x*-*y* ***plane***.

As for usual functions, determining the ***domain*** of a two-variable functions ***can usually be done***, unless the function has some unusual algebraic properties, while the ***range is more difficult*** to determine and can only be identified if the function is defined by a sufficiently simple formula.

Since a two-variable function has two inputs and one output, its graph will be a surface, such as the one shown here, but studying the graphical features of such surfaces is beyond our goals and you will see that in future courses.



Instead, for now we shall focus on those basic aspects of these functions that are related to our main topic of regular, one-variable functions. The next concept provides just one such connection.

Definition

A ***level curve***, or ***level set***, of a two-variable function $z = f(x, y)$ is the two dimensional curve obtained by assigning to z a constant value. That is, a level curve is the set of points in the x-y plane whose coordinates satisfy an equation of the form $f(x, y) = c$.

Example: $z = \sqrt{x - y}$

We can get a level curve for this function by assigning to z any non-negative constant value. For instance, the following are level sets for it:

$$\sqrt{x-y} = 4 \quad ; \quad \sqrt{x-y} = 0 \quad ; \quad \sqrt{x-y} = \pi \quad \text{etc.}$$

In this simple case, we can get a better understanding of each such curve by changing its equation to a more informative form, as follows:

$$\sqrt{x-y} = c \quad \Leftrightarrow \quad x - y = c^2 \quad \Leftrightarrow \quad y = x - c^2$$

This tells us that each level curve is a line with slope of 1 and non-positive intercept.

Example: $z = x^2 + \sin y$

This time the level curves have equations of the form:

$$x^2 + \sin y = c$$

These are not regular functions and need to be analyzed individually.

This last example brings up a point that will be used frequently in the future.

Knot on your finger

Each level curve of a two-variable function determines an ***implicit relation*** between the variables x and y. Such relation may, in some situations, define a regular function $y = f(x)$, but, more generally, provides a relation that may be studied with suitable variations of calculus methods.

Before moving on, I want to point out that there is no reason to restrict our attention to only *two* input variables. We can consider functions that have three or more input variables, and they are useful in their own right. However, for the time being we shall stick only to the two-variable case.

Summary

- A two-variable function uses two input values to generate an output value.
- If we fix the output value of a two-variable function, we obtain a level curve, which, in general, is an implicit relation between the two input variables.

Common errors to avoid

- Do not confuse a function of two variables with a function of one variable that also includes an unspecified constant.

Learning questions for Section P 4-1

Review questions:

1. Explain what a two-variable function is.
2. Describe what a level curve of a two-variable function is.

Memory questions:

1. How many independent variables does a two-variable function have?
2. How many dependent variables does a two-variable function have?
3. What constitutes the domain of a function $z = f(x, y)$?
4. If $f(x, y)$ is a two-variable function, what is the name of a curve described by $c = f(x, y)$?

Computation questions:

For the function provided in each of questions 1-14, determine:

a) its domain
b) any limitations on the range, if any, that can be identified algebraically
c) the value of the function at a few pairs of numbers of your choice
d) the level curves indicated, expressed, if possible, as usual functions.

1. $f(x, y) = \dfrac{x + xy}{y - 1}$, $z = 1$
2. $f(x, y) = \dfrac{x^2}{y + 1}$; $z = -2$
3. $f(x, y) = \dfrac{\sqrt{x - y}}{x + y}$; $z = 1$

4. $f(x,y)=x^2\sqrt{3-y}$; $z=3$

5. $f(x,y)=\dfrac{x^2}{\sqrt{y^2-xy}}$; $z=2$

6. $f(x,y)=y^2-\sqrt{xy}$; $z=2$

7. $f(x,y)=\cosh(xy^3)$, $z=1$; $z=\dfrac{e+e^{-1}}{2}$

8. $g(x,y)=\ln(x^2+y)$; $z=0$

9. $f(x,y)=\sin(x^3y)$, $z=\dfrac{1}{2}$

10. $f(x,y)=\cos(x^3y)$, $z=-1$

11. $f(x,y)=3y+\cos x$; $z=a$

12. $g(x,y)=\cos(x+y)$; $z=2$

13. $f(x,y)=x^4+y^3-x^2+1$; $z=1$

14. $f(x,y)=x^4+y^5-x^2+1$; $z=0$

For each pair of two-variable functions provided in questions 14-20, compute:

a) $f(x,y)+g(x,y)$ b) $\dfrac{g(x,y)}{f(x,y)}$ c) $f(x,g(x,y))$ d) $g(f(x,y),x)$

15. $f(x,y)=\dfrac{x-xy}{y-1}$; $g(x,y)=\cosh(xy^3)$

16. $f(x,y)=\dfrac{x^2}{y+1}$; $g(x,y)=\ln(x^2+y)$

17. $f(x,y)=x^2\sqrt{3-y}$; $g(x,y)=\sin(x^3y)$

18. $f(x,y)=\dfrac{\sqrt{x-y}}{x+y}$; $g(x,y)=\cos(x^3y)$

19. $f(x,y)=\dfrac{x^2}{\sqrt{y^2-xy}}$; $g(x,y)=y^2-\sqrt{xy}$

20. $f(x,y)=3y+\cos x$; $g(x,y)=\cos(x+y)$

21. Find the level curve $z=0$ for the function $f(x,y)=xy-\cosh^{-1}(e^{xy})$. Additionally, either sketch the level curve or describe what it looks like. Please notice that while it will be difficult to tell immediately what the level curve looks like, it will become more obvious after you have made enough simplifications to the algebraic expression which defines this curve.

Theory questions:

1. Can $z = x^3$ be considered as a function of two variables?

2. What kind of relation between x and y does a level curve provide?

3. Can the level curve of a function of two variables be reduced to a single point?

Application questions:

1. Tangent Lake is a square lake whose depth at a point x km East and y km North from the SW corner is given by the function $h(x, y) = \frac{\sin x}{\cos y}$, where

   $0 \le x \le \pi, 0 \le y \le \pi / 4$. Determine:

   a) The depth of the lake at the middle of the South shore.
   b) The point with greatest depth.
   c) The function representing the curve at which the depth is half a km.

Templated questions:

1. Construct a level curve for any two-variable function you use.

2. Determine the domain of any two-variable function you can come up with.

What questions do you have for your instructor?