Chapter 4: Special types of functions *Section 2*

Parametric curves

What you need to know already:

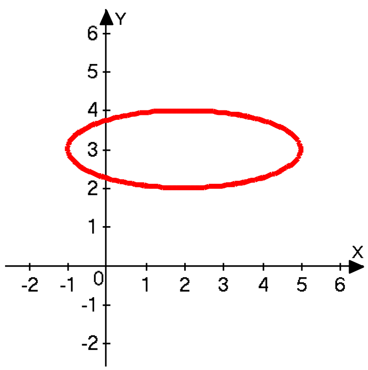
- Functions and their graphs.

What you can learn here:

- How to describe a curve in a richer way by including a parameter.

A usual function is a relation between an input variable x and an output variable y that satisfies the vertical line test: for each x value there is at most one y value. But not all relations, and not all curves, satisfy this requirement. Moreover, the graph of a function is a somewhat static picture, even when it is used to describe the movement of an object or the development of a quantity.

For instance, this ellipse is not the graph of a function, but it may be the trajectory of a planet moving around the Sun. So, how can we represent this curve mathematically, as well as its additional dynamic features?



In fact, the key idea is indeed to think of a curve as the trajectory of a moving object and to describe the x and y coordinates as separate functions of t, except that t does not necessarily have to represent *time* but can be any generic additional variable.

This additional variable t is called a *parameter* and it can range over any specified interval, or even over all real numbers.

Definition

A ***parametric curve*** in $\mathbb{R}^2$ is a set of points in the Cartesian plane whose coordinates (x, y) are described by two ***parametric functions***:

$$x = x(t), \quad y = y(t)$$

For this reason, a parametric curve can be denoted as above, or as a pair of coordinates:

$$\left(x(t), y(t)\right)$$

The point $\left(x(0), y(0)\right)$, if it exists, is called the ***starting point*** of the curve.

The tracing direction of the curve that corresponds to increasing values of t is called the ***orientation*** of the parametric curve.

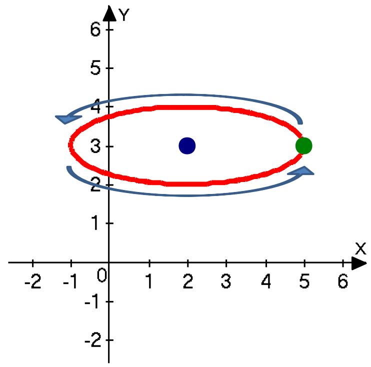
Example: $\left(2+3\cos t, 3+\sin t\right)$

It turns out that these parametric functions describe the ellipse presented at the beginning of the section. You may want to check this on your graphing calculator, by putting it in parametric mode.

Notice that the starting point for this curve is:

$$\left(2+3\cos 0, 3+\sin 0\right)=\left(5,3\right)$$

This is the point on the extreme right of the ellipse.



As we let t increase, we notice that the first coordinate, which involves the cosine function, will decrease, while the second coordinate, which involves a sine function, increases. This tells us that the curve is traced counter-clockwise: this is its *orientation*.

As an added bonus, in this case we see from the formula that the centre of the ellipse is at the centre of the oscillations of the two coordinates, that is, at $\left(2,3\right)$.

Can you appreciate how much richer this formulation is than the traditional $y=f\left(x\right)$?

Will we always use the calculator to get the graph of a parametric curve?

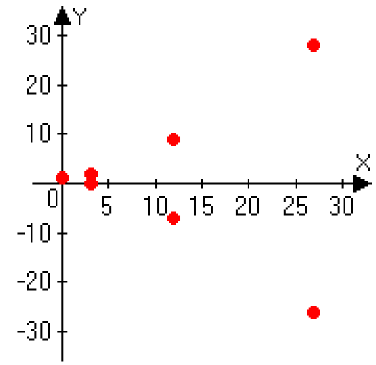
You can also use a table of values approach: compute the values of x and y for several values of t and then plot and join the resulting points. This is a method analogous to what you first used to sketch the graph of a function, only with the additional parameter to be used.

Example: $\left(3t^2, 1-t^3\right)$

By using several small and easy values of t, we can construct the following table of values:

t	x	y
-3	27	28
-2	12	9
-1	3	2
0	0	1
1	3	0
2	12	-7
3	27	-26

By plotting these values, we obtain the following picture:



Is the graph really as simple as what this shows, or are we missing some important features? We can add more points, but for what values of t?

Use a calculator to investigate this further.

As you can see, this method involves a tedious task that often provide little or no convincing information about the graph, so yes, we shall use the calculator for now. Later you shall learn calculus methods aimed at more clearly and safely identifying important graphical details of a parametric curve.

However, keep in mind that when you use your calculator in parametric mode, you have to decide on the window limits not only for x and y, but for t as well. Choosing the wrong values for t may produce an incorrect or incomplete graph. Play with that to learn how to do it properly.

So do we use parametric curves only to study curves that are not functions?

Primarily yes, but not exclusively. There are advantages to representing even a usual function in parametric form, given the additional information provided by this form.

The following considerations may help you better understand the advantages of using parametric functions, even for regular functions.

Knot on your finger

Any ***function*** $y = f(x)$ can be written in ***parametric form*** by letting $t = x$:

$$y = f(x) \quad \Leftrightarrow \quad (t, \quad f(t))$$

However, the same function can be written in parametric form in infinitely many other ways, by letting $x = g(t)$ for some function *g*:

$$y = f(x) \quad \Leftrightarrow \quad (g(t), \quad f(g(t)))$$

Depending on the function $g(t)$, this may allow for a different orientation, speed of tracing, or even the elimination of some sections of the graph.

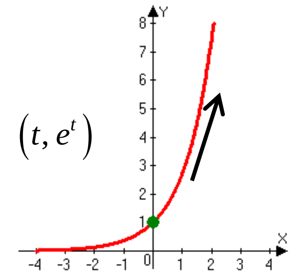
I am unclear: what are you trying to say?

That we can use the parametric approach to describe the graph of a function in different ways that may suit the particular application we may have in mind. I hope this example clarifies things a bit.

Example: $y = e^x$

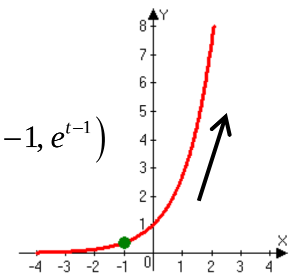
We can write this function in parametric form in the standard way, as:

$$y = e^x \quad \Leftrightarrow \quad (t, e^t)$$



In this way the starting point is the *y* intercept $(0,1)$ and the orientation is from left to right. But we can describe the same curve with a different parametrization by letting $x = t - 1$. In this way the formula becomes:

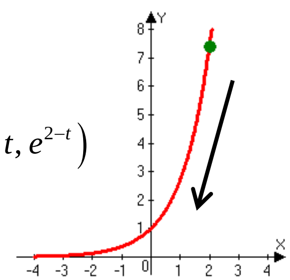
$$y = e^x \quad \Leftrightarrow \quad (t-1, e^{t-1})$$



Now the starting point is $\left(-1, \frac{1}{e}\right)$ and the direction is still left to right.

We can try yet another approach. If we let $x = 2 - t$, we get:

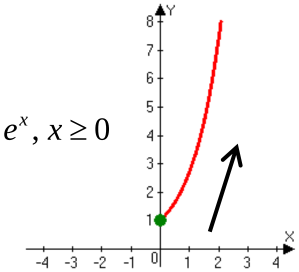
$$y = e^x \quad \Leftrightarrow \quad (2-t, e^{2-t})$$



In this way the starting point is $(2, e^2)$ and the orientation is right to left.

Or we can let $x = t^2$:

$$\left(t^2, e^{t^2}\right) \Leftrightarrow y = e^x, x \geq 0$$



This uses the same original starting point and orientation, but it describes only the portion of the function that is in the second quadrant.

Try all these, and maybe more, on your graphing calculator to see the effect.

Since we can write any function in parametric form, as a pair of equations, can we do the opposite? That is, can we identify a parametric curve by a single equation?

As you can imagine, this cannot always be done, since going from two to one equation implies loss of information. However, it can be done in some cases and it can be a revealing exercise.

Definition

The process of ***eliminating the parameter*** from a parametric curve $\left(x(t), y(t)\right)$ consists of expressing the curve through a single equation of the type $f(x, y) = 0$, that is as a level curve of a two-variable function.

Eliminating the parameter is not always possible or easy and it does not always lead to an explicit function, as when the curve is not the graph of a function. Also, this process looses the starting point and orientation. But in some instances this reformulation may reveal some interesting features of the curve or may allow us to apply certain calculus procedures in an easier way.

Example: $(2 + 3\cos t, 3 + \sin t)$

We have noticed that these parametric functions describe an ellipse, but an ellipse can be written through a single Cartesian equation of the form:

$$\frac{(x-a)^2}{p^2} + \frac{(y-b)^2}{q^2} = 1$$

In this form, the centre of the ellipse is at (a, b) and we know that the horizontal "*diameter*" is long $2p$ and the vertical one is long $2q$. This may be useful in some cases, so can we change the parametric form to this Cartesian form?

To do that, we isolate the sine and cosine portions of the parametric function:

$$x = 2 + 3\cos t,\ y = 3 + \sin t \quad \Rightarrow \quad \cos t = \frac{x-2}{3}, \sin t = y - 3$$

Now we use the fundamental trigonometric identity:

$$\cos^2 t + \sin^2 t = 1 \quad \Rightarrow \quad \left(\frac{x-2}{3}\right)^2 + (y-3)^2 = 1$$

$$\Rightarrow \quad \frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{1^2} = 1$$

And we have the Cartesian form.

Is this a worthwhile exercise? I am not sure in this case, but it is a method that you need to know for later applications.

Example: $\frac{x^2}{3} - \frac{y^2}{4} = 5$

We can obtain a parametric expression for this curve by using the fact that:

$$\sec^2\theta - \tan^2\theta = 1$$

If we divide both sides of our equation to 1, we get:

$$\frac{x^2}{3} - \frac{y^2}{4} = 5 \quad \Rightarrow \quad \frac{x^2}{15} - \frac{y^2}{20} = 1$$

This is similar to the trig identity, so that we can set:

$$x = \sqrt{15}\sec\theta,\ y = \sqrt{20}\tan\theta$$

We now have parametric equations that satisfy the original relation, hence they are the ones we seek.

As we saw in a previous example, we can also devise a parametric version of a curve to identify just a piece of the graph of a regular function. Although this goal can also be achieved by simply restricting the domain, there are situations in which doing it the parametric way can be useful.

Example: $\left(\frac{2}{t}, \frac{8}{t^2}\right)$

If we eliminate the parameter, we see that this is just a parabola:

$$x = \frac{2}{t} \quad \Rightarrow \quad y = 2\left(\frac{2}{t}\right)^2 = 2x^2$$

But notice that this parametric version does not include the origin! A neat trick to exclude a point, no?

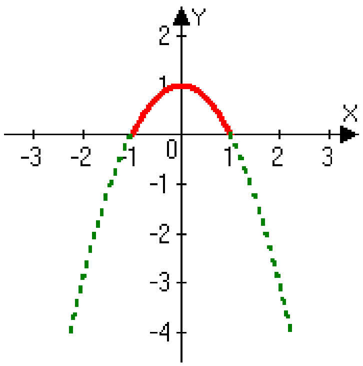
And of course, this applies when the section of the curve represents the trajectory of an object that covers the same section repeatedly: remember that the parametric representation of a curve includes both orientation and speed.

Example: $\left(\sin t, \cos^2 t\right)$

Again by eliminating the parameter we obtain the equation of a parabola:

$$x = \sin t \quad \Rightarrow \quad y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$$

But this parametric version only works as long as $-1 \le x \le 1$ and $0 \le y \le 1$ because of the range of the sine and cosine function. This means that we are not describing the whole parabola, but only the section of it above the horizontal axis.



Not only that, but that small section is traced over and over, because of the periodic nature of the sine and cosine functions.

Summary

- Parametric functions can be used to identify a curve that is not the graph of a function or to add features to the graph of a function.
- A parametric form can be used to add a starting point and orientation, or to limit the domain of the curve.
- In some cases, but not all, the parametric form of a curve can be changed to a single Cartesian equation, if that provides useful information about the curve.

Common errors to avoid

- When using your graphing calculator to see a parametric curve, set the window values for t properly, or you may end up with and incomplete or incorrect graph that may lead you to answer incorrectly any questions about the curve.

Learning questions for Section P 4-2

Review questions:

1. Explain what a parametric curve is and how it is denoted.
2. Identify the main advantages of describing a curve in parametric form.
3. Explain what is meant by *eliminating the parameter* and identify some methods for doing so to a parametric curve.

Memory questions:

1. How many functions define a parametric curve?
2. How many independent variables does a parametric function have?
3. How many dependent variables does a parametric function have?
4. What are the parametric equations of a function $y = f(x)$?
5. Which is the *starting point* of a parametric curve $(x(t), y(t))$?
6. What is the name of the process by which we change a parametric function into a single equation?

Computation questions:

For each of the parametric curves provided in questions 1-32:

a) construct a table of values and use it to obtain a sketch of its graph;
b) use your graphing calculator to obtain the graph of the curve in several windows;
c) identify starting point and orientation;
d) explain as many graphical features as you can by using algebraic properties of the functions that define the curve;
e) if possible, eliminate the parameter to obtain a Cartesian formula for this curve;
f) construct a different parameterization for the same curve.

1. $\left(2t, 4t^2\right)$

2. $\left(t^2,\ t^3-3t\right)$

3. $\left(3t^2, 1-t^3\right)$

4. $\left(-t^2+t+1, t^3-3t+1\right)$

5. $\left(\dfrac{t^3-t}{4}, t^2-3\right)$

6. $\left(t^3, t^2-6t\right)$

7. $\left(\sqrt{t^2-1}, \sqrt{t^2+1}\right)$

8. $\left(\sqrt{t-5}, e^{\sqrt{t}}\right)$

9. $\left(e^t, t^3-t^2\right)$

10. $\left(e^t, e^{-t}\right)$

11. $\left(1+3e^t, 2-e^{2t}\right)$

12. $\left(2\ln t+4, e^2 t\right)$

13. $\left(\ln t, \dfrac{t}{t-2}\right)$

14. $\left(\ln 2t, \dfrac{2}{t-2}\right)$

15. $\left(\cosh t, \sinh t\right)$

16. $\left(\tanh t, \operatorname{sech} t\right)$

17. $\left(\tan t, 2\sec t\right)$

18. $\left(\cos t, 2\sin t\right)$

19. $\left(\sin t, \cos 2t\right)$

20. $\left(\cos 5t,\ \sin 3t\right)$

21. $\left(\cos \pi t,\ \sin 3t\right)$

22. $\left(t-3\sin t, 1-3\sin t\right)$

23. $\left(t-\sin t,\ 1-\cos t\right)$

24. $\left(t\sin t,\ t\cos t\right)$

25. $\left(\sin t - t\cos t, \cos t + t\sin t\right)$

26. $\left(\dfrac{\cos t}{t}, \dfrac{\sin t}{t}\right)$

27. $\left(2\cot t,\ 2\sin^2 t\right)$

28. $\left(\sec t, 3\tan t\right)$

29. $\left(\sec t, \tan^2 t\right)$

30. $\left(3+\cosh^2 t, 2-\sinh t\right)$

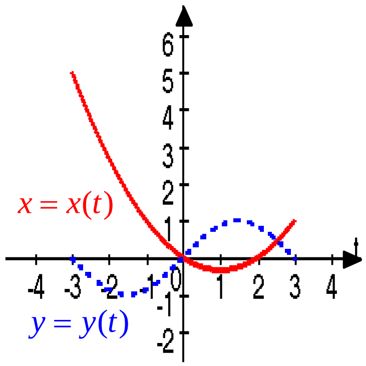
31. $\left(3+\sinh t, 2-\cosh^2 t\right)$

32. $\left(\tan t, \cos^2 t\right)$ (This is called a Witch of Agnesi)

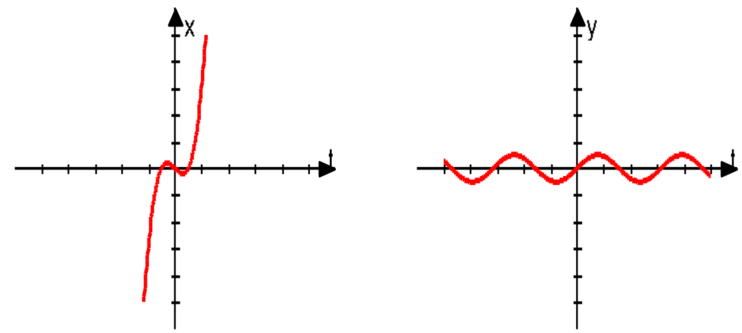
33. Find the coordinates of the point(s) of intersection of the curves $\left(\ln t^2, \ln t\right)$ and $x^2-2y^2=1$.

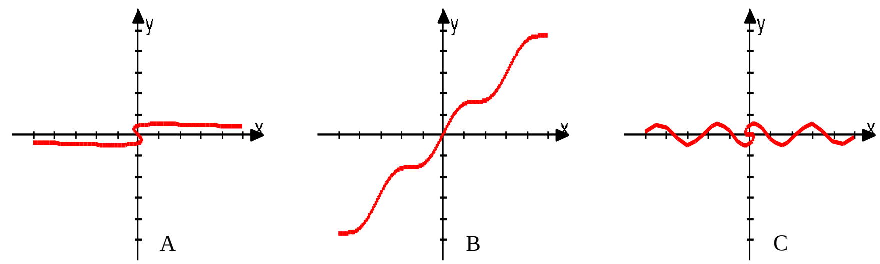
34. Find all intersection points of the parametric curve $\left(e^t, 10e^{-t}\right)$ and the circle $x^2+y^2=29$.

35. A parametric curve $(x(t), y(t))$ is defined by the functions $x = x(t), y = y(t)$ shown below. Use these graphs to sketch a graph of the parametric curve and briefly describe the method you use to arrive at your conclusion.



36. The graphs of the functions $x(t)$ and $y(t)$ are as shown here. Identify which of the three graphs given below them represent the parametric curve $(x(t), y(t))$ and provide two arguments in support of your choice and one argument to eliminate each of the other possible choices.





37. Construct parametric equations of the hyperbola $x^2 - 5y^2 = 3$.

Theory questions:

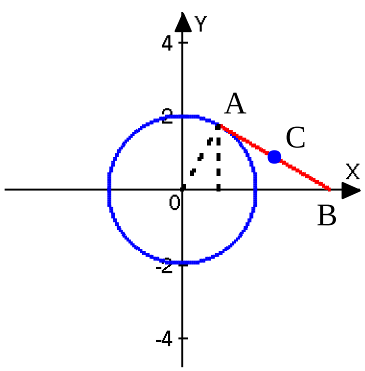
1. For which curves is a parametric representation necessary?
2. For which curves is a parametric description most useful?
3. When is it possible to eliminate the parameter from a parametric curve: always, sometimes or never?
4. Mention two features of the graph of a curve that can be better controlled by using a parametric approach.
5. What is the main difference between a parametric curve and a two-variable function?
6. What is the easiest way to change the formula of a parametric curve so that its orientation is reversed?

Proof questions:

1. What are the parametric equations of a circle centered at the origin and with radius *r*?
2. Explain why the parametric function $\left(t\sin t, e^{t+1}\right)$ never crosses the *x*-axis and determine what happens to the graph of this function when *t* approaches -∞. Again, pure calculator work is not what I expect.

Application questions:

1. Given a point A on the circle of radius 2 centered at the origin, consider the tangent at that point and let B be the point where this tangent crosses the *x* axis (if it does). Express the coordinates of the midpoint C of the segment AB in terms of the angle $\theta = BOA$ and explain why these describe a parametric curve. This is also a [Witch of Agnesi,]



2. Use your calculator to sketch the graph of the parametric curve given by the functions

$$x = (1-t)^3 + 3t(1-t)^2 + 3.3t^2(1-t) + 2t^3$$

$$y = 4(1-t)^3 - 9t(1-t)^2 + 15t^2(1-t) + t^3$$

For $0 \le t \le 1$. This is a rudimentary way to set up computerized fonts.

Templated questions:

1. For each parametric curve used in another question, construct a different pair of equations that represents the same curve travelled:
   a) at a different speed;
   b) with a different starting point;
   c) with a different orientation.

2. For any parametric curve that describes a function, determine the domain of the function.

3. Write any usual function in standard parametric form and then in a different parametric form.

What questions do you have for your instructor?