Chapter 4: Special types of functions *Section 4*

Polar curves

What you need to know already:	*What you can learn here:*
➤ Polar coordinates and the concept of graph of a function.	➤ How to use polar coordinates to described curves that are suitable for them.

Since polar coordinates are based on a fixed point (the pole) and angles around it, they are particularly useful when dealing with curves that have some rotating pattern around the origin.

In that case it may be convenient to describe such a curve by relating the polar coordinates, rather than the Cartesian ones.

Definition

A ***polar curve*** is the set of points whose polar coordinates satisfies an equation of the form:

$$r = f(\theta)$$

In mathematics we try to avoid unnecessary writing, and here we have another instance of how this is done.

Knot on your finger

Usually, the function that defines a polar curve is denote by the letter *r*, as $r = r(\theta)$.

By using the formulae that change polar to rectangular coordinates, we can write any polar curve in parametric form.

Knot on your finger

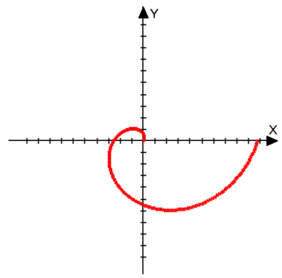
Any polar curve can be written in parametric form by using θ as the parameter:

$$r(\theta) \quad \leftrightarrow \quad \left(r(\theta)\cos\theta,\ r(\theta)\sin\theta\right)$$

As for parametric curves, we can start exploring polar curves by using a graphing calculator in polar mode. And, as for parametric curves, we need to be careful to:

- set the window values correctly, so that the whole curve is clearly visible, and
- check that any relevant feature of the graph can be explained by using algebraic properties of the function and of polar coordinates, lest we become fooled by some calculator glitch.

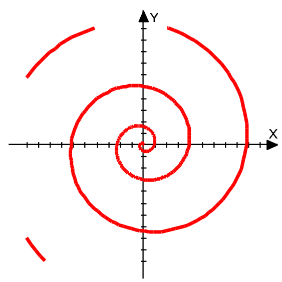
Example: $r = \dfrac{1}{4}\theta^2$



If you try getting this graph in the standard calculator window, you will likely get a graph like this one. Nice picture, but does it tell the whole story?

If we think about what is happening, we realize that the calculator is short-changing us on two counts:

- The default values of $0 \le \theta \le 2\pi$ offer us a single rotation around the pole, while the function can be evaluated for larger values of *θ*.
- The default values of $-10 \le x, y \le 10$ do not allow us to see what happens for larger values of θ, since the square function will then produce large values or r, that is, the curve will be far from the pole.

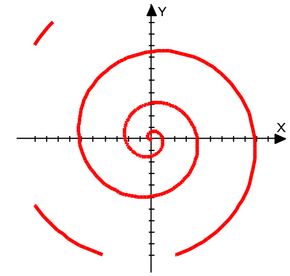


If we adjust both windows, by letting $0 \le \theta \le 8\pi$ and $-100 \le x, y \le 100$, we get this much more informative picture.

We can see that we are dealing with a spiral that becomes progressively wider and farther from the pole.

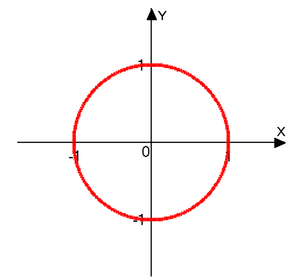
But there is something still missing. What about negative values of *θ*?

By using $-8\pi \le \theta \le 0$, we get this graph: a spiral in the opposite direction. You may want to try using $-8\pi \le \theta \le 8\pi$: you will see an interesting combination of spirals!



As you can imagine, that paragon of rotation, a circle with radius 1, has a very simple polar form, namely:

$$r = 1$$



The same can be said of any circle centered at the pole.

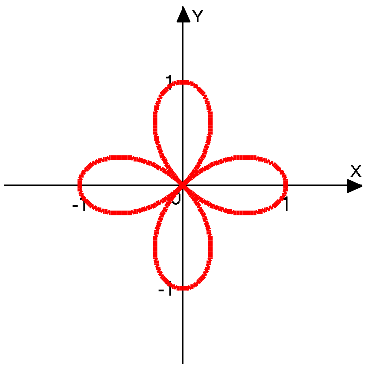
There are many more intriguing curves that can be obtained by using polar functions. I will leave most of them for the *Learning Questions*, but I do want to mention one type that will be used extensively once we develop calculus methods for polar curves.

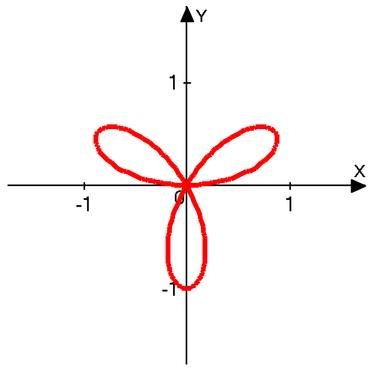
Definition

The graph of a polar function of the form $r = \cos n\theta$ or $r = \sin n\theta$, where *n* is a positive integer, is called a ***rose***, even though it more closely resembles a daisy ☺.

Example:

Here are $r = \cos 2\theta$, on the left, and $r = \sin 3\theta$ on the right.





Play with different values of *n* and see if you can figure out the connection between it and the number of petals of the rose, as well as the location of the petals, depending on which function is used.

Since a polar curve can always be written as a parametric curve, we may wonder if it is possible to eliminate the parameter θ. Of course it can, but not always and not always easily.

Here is one example of how it can be done.

Example: $r = \dfrac{4\sin^2\theta + 3}{\cos\theta}$

To obtain the Cartesian equation of this polar curve, we first eliminate the denominator:

$$r = \frac{4\sin^2\theta + 3}{\cos\theta} \quad \Rightarrow \quad r\cos\theta = 4\sin^2\theta + 3$$

Notice that the left side simply describes the *x* coordinate $(x = r\cos\theta)$, so we can look for a way to change the right side. From the formula $y = r\sin\theta$ we see that:

$$\sin\theta = \frac{y}{r} \quad \Rightarrow \quad \sin^2\theta = \frac{y^2}{r^2} = \frac{y^2}{x^2 + y^2}$$

This tells us that the original equation may be written as:

$$r\cos\theta = 4\sin^2\theta + 3 \quad \Rightarrow \quad x = \frac{4y^2}{x^2 + y^2} + 3$$

Finally, remember that polar functions are well suited for curves that have a rotating pattern around the origin. We can use them to describe lines, as you will see in the *Learning Questions*, but there is no advantage to doing that, besides the computational practice!

Summary

- A polar curve is the graph of a curve whose equation, in polar coordinates, is of the form $r = f(\theta)$.

Common errors to avoid

- Watch out for the window values when using the graphing calculator!

Learning questions for Section P 4-4

Review questions:

1. Describe what a polar curve is and how it is usually provided algebraically.

2. Describe how to change the formula of a polar curve into Cartesian form and vice versa.

Memory questions:

1. For which curves are polar coordinates appropriate?

2. Which polar function describes a circle with center at the pole and radius *k*?

3. What are the parametric equations of a polar curve $r = f(\theta)$?

Computation questions:

Sketch a graph of each of the polar curves provided in questions 1-12, first by hand, by plotting at least 6 points in polar coordinates, and then by using a graphing calculator.

1. $r = \dfrac{2\theta}{\pi}$

2. $r = \dfrac{2\pi}{\theta}$

3. $r = e^{-\theta}$

4. $r = \ln\theta$

5. $r = \sin(n\theta)$, $n = 2, 3, 4, 5, 6$

6. $r = \cos(n\theta)$, $n = 2, 3, 4, 5, 6$.

7. $r = 3\sin 2\theta$

8. $r = 2\cos 3\theta$

9. $r = 3 - 2\sin\theta$

10. $r = 2 - 6\sin\theta$

11. $r = \cos^2\theta + 2 - \sin\theta$

12. $r = \cos^2\theta + 2$

In each of questions 13-20 you are given a Cartesian function and the same function as a polar function. Use your calculator to obtain the graphs of both functions and describe the results in terms of some algebraic properties of the function.

13. $y(x)=3$; $r(\theta)=3$

14. $y(x)=x$; $r(\theta)=\theta$

15. $y(x)=x-2$; $r(\theta)=\theta-2$

16. $y(x)=x^2$; $r(\theta)=\theta^2$

17. $y(x)=(x-1)^2$; $r(\theta)=(\theta-1)^2$

18. $y(x)=x^2-1$; $r(\theta)=\theta^2-1$

19. $y(x)=(x-1)(x+2)$; $r(\theta)=(\theta-1)(\theta+2)$

20. $y(x)=x^2-4x+8$; $r(\theta)=\theta^2-4\theta+8$

21. For what value of θ does the graph of the polar curve $r=1+4\cos(3\theta)$ cross the pole?

22. For what value of θ does the graph of the polar curve $r=1+4\cos(3\theta)$ cross the polar axis?

23. What values of θ produce the intersections of the curves $r=2\sin(3\theta)$ and $r=1$?

24. What are the parametric equations of the curve $r=2\sin(3\theta)$?

25. Determine the single Cartesian equation of the polar curve $r^3=4\csc\theta$ and explain two of the graphical features visible in a calculator graph by identifying the corresponding algebraic explanation.

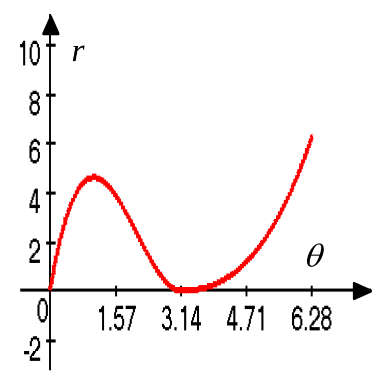
26. Obtain a single Cartesian equation that describes the polar curve

$$r=\frac{2}{1+2\cos\theta}$$

27. Which polar function describes the curve defined by the Cartesian equation $x^2-y^2=x$?

28. Which polar function describes the curve defined by the Cartesian equation $\ln(x^2+y^2)=5+3\ln(x-y)$? Make sure to solve for $r(\theta)$ explicitly.

29. A function $r=f(\theta)$ is represented by the graph shown here in Cartesian coordinates.



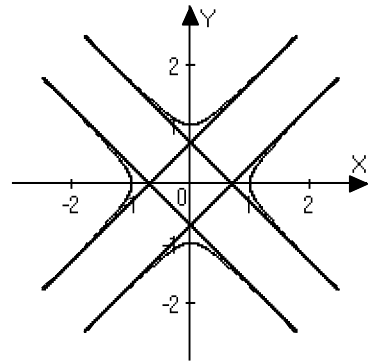
Use this graph to sketch the graph of the same function when viewed as a polar function and briefly explain how you obtained such graph.

30. Provide a valid algebraic argument explaining why the polar curve $r = \cos^2\theta + 2 - \sin\theta$ is entirely within the circle $x^2 + y^2 = 16$.

31. Determine the formula in polar coordinates for the line described by the Cartesian equation $2x + y = 4$ and show that the angle θ formed by this line with the x-axis is outside the domain of the polar function.

32. Provide a valid algebraic argument explaining why the polar curve $r = \cos^2\theta + 2$ is entirely within the circle $x^2 + y^2 = 9$.

33. If you sketch the graph of the polar function $r = \dfrac{1}{\cos 2\theta}$ in the standard window of a graphing calculator, and then zoom in once, you may get a graph similar to the one shown here.



a) Provide a mathematical argument explaining why this graph cannot be correct
b) Explain why the calculator makes such an error
c) Use a suitable double angle formula to obtain a Cartesian equation describing the same curve.

34. Use your calculator to determine the graph of the *"Devil's curve"*: $r = \sqrt{\dfrac{25 - 24\tan^2\theta}{1 - \tan^2\theta}}$. Be careful, as a single attempt may provide a misleading graph!

Theory questions:

1. Every polar function can be considered as a parametric function. What does its parameter represent?

2. How many parametric functions exist that represent a given polar curve?

3. Is it possible for a polar curve to go through the same point for opposite values of θ?

4. Which curve does the polar equation $\theta = 2$ represent? Be as specific as possible.

5. Why is it not convenient to represent the function $y = \sin x$ as a polar curve?

6. Why does the polar curve $r = 2 - 3\cos\theta$ cross the pole, while the curve $r = 3 - 2\cos\theta$ does not?

Proof questions:

1. Prove that the polar curve $r = \dfrac{1}{1-\cos\theta}$ is a parabola.

2. Prove that the polar curve $r = \dfrac{1}{\cos\theta - \sin\theta}$ is a line.

3. Use the parametric equations of a polar curve to explain why the polar curves $r = \cos 4\theta$ and $r = -\cos 4\theta$ produce the same graph.

4. Prove that the polar curve $r = \cos\theta$ is a circle.

Templated questions:

1. Write any polar curve you choose in parametric form.

2. Use your calculator to find the graph of the polar curve $r = \cos k\theta$ for any real value of *k* that you choose and try to determine any pattern features related to the choice of *k*.

What questions do you have for your instructor?