Chapter 1: Limits and continuity — *Section 2*

Graphical and numerical methods
to estimate limits

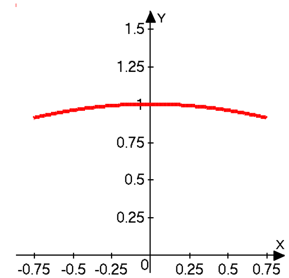
What you need to know already:	*What you can learn here:*
➤ Definition and notation for limits.	➤ Two practical ways to obtain an estimate of the value of a limit.

When confronted with the need to compute a limit, our ultimate goal is to compute its exact value and we shall soon see several methods for doing just that. However, in many situations, and especially at this stage of the game when you are still getting familiar with limits, there are two practical ways to get a quick estimate of the value of the limit, or to get some evidence for its non-existence.

Definition

The ***graphical*** method of estimating a limit consists of using a graphing calculator or computer program to observe visually the behaviour of the function near the given value and thus making an estimate of the limit value, if it exists.

Example: $\lim_{x \to 0} \frac{\sin x}{x}$



This is an interesting limit, as both numerator and denominator become 0 at $x = 0$. The graph of the function for values of *x* near 0 is as shown here and it gives us the distinct impression that the limit is 1, or at least a number very close to 1.

This observation gives us an estimate that is useful, even though it needs to be confirmed with other more accurate and reliable methods.

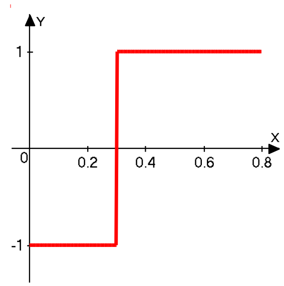
That's easy!

Don't get too excited by the apparent ease with which this method seems to provide answers. An excessive reliance on this method can turn into a trap.

Warning bells

While useful, the graphical method provides only an ***estimate*** and sometimes it may provide incorrect conclusions.

Example: $\lim\limits_{x \to 0.3} \dfrac{\sqrt{(10x-3)^2}}{10x-3}$



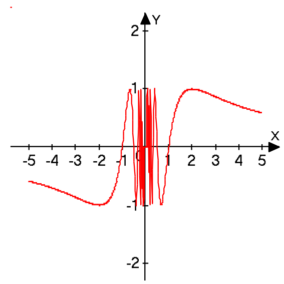
A very good computer program produced this graph for this limit. It seems to suggest that the limit we are looking for is 0, since the graph seems to have an x intercept at $x = 0.3$. But this is not true, as we shall see soon.

Notice that the graph also seems to suggest that:

$$\lim_{x \to 0.3^-} \frac{\sqrt{(10x-3)^2}}{10x-3} = -1 \quad ; \quad \lim_{x \to 0.3^+} \frac{\sqrt{(10x-3)^2}}{10x-3} = 1$$

So, what is the truth? Is the function approaching 0, 1, -1 or none of them? We need better methods to resolve such indecisions.

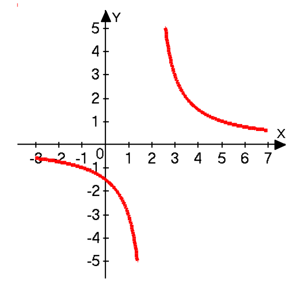
Example: $\lim\limits_{x \to 0} \sin \dfrac{\pi}{x}$



If you look at this graph, can you arrive at an estimate for this limit? Will the visual effect improve if we use a smaller window, thus focusing on *x* values closer to 0? You may want to try.

Is it possible that this limit does not exist? We cannot be sure, based on the graphical method only.

Example: $\lim\limits_{x \to 2} \dfrac{3}{x-2}$



What can we conclude about this limit? We seem to be missing a portion of the graph, just near the value we want: where did it go? Can we change the window in order to retrieve it? Try that.

Can we conclude that the limit does not exist? Could it be that the function will become infinite? All good hypotheses, but the graph is not convincing.

Definition

The ***numerical*** method of estimating a limit consists of constructing a table of values for the function and for values of *x* closer and closer to *c*. The pattern of *y*-values so obtained can be used to generate an estimate for the limit.

Warning bells

While useful, the numerical method also provides only an estimate and, sometimes an incorrect conclusion.

Example: $\lim_{x\to 0}\frac{\sin x}{x}$

If we use the numerical method for this limit, we may end up with this table of values:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
y	.99833	0.99998	0.99999	0.99999	0.99998	0.99833

This table also suggests that the limit is 1, but are we sure it is exactly 1? Could it be just a little below 1? Again, this method gives an estimate and possibly a lot of confidence, but no exact value.

Example: $\lim_{x\to 0.3}\frac{\sqrt{(10x-3)^2}}{10x-3}$

A table of values provides the following:

x	0.29	0.299	0.2999	0.3001	0.301	0.31
y	-1	-1	-1	1	1	1

This table also seems to suggest, and even more strongly, that:

$$\lim_{x\to 0.3^-}\frac{\sqrt{(10x-3)^2}}{10x-3}=-1 \quad ; \quad \lim_{x\to 0.3^+}\frac{\sqrt{(10x-3)^2}}{10x-3}=1$$

Can we be sure now? What if we get different values by getting even closer to 0.3? We have better evidence, but still no certainty.

And now, for something quite sobering. There are some limit situations for which the numerical method can suggest a very convincing conclusion that is, however, wrong! For instance, the seemingly simple example that follows.

Example: $\lim_{x\to 0}\sin\frac{\pi}{x}$

If we try the same method we get the following:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
y	0	0	0	0	0	0

Well, the pattern seems clear: the limit is 0, right? Wrong! Try different tables, by using different values of x getting close to 0 and you will find something quite interesting!

Example: $\lim_{x\to 2}\frac{3}{x-2}$

Same method:

x	1.9	1.99	1.999	2.001	2.01	2.1
y	-30	-300	-3000	3000	300	30

It seems that the function is growing bigger and bigger as x approaches 2, with negative values on the left and positive on the right. What do we make of that? Do we dare say that:

$$\lim_{x\to 2^-}\frac{3}{x-2}=-\infty \quad ; \quad \lim_{x\to 2^+}\frac{3}{x-2}=\infty$$

Do we even know what that means? It is time to look for more accurate methods.

Summary

- Graphs and tables obtained from calculators may be used to get an estimate of the value of an interesting limit.
- Both methods can provide reasonable and even accurate guesses for the value of the limit.
- However, both methods are approximate in nature and can lead to incorrect conclusions if used with too little care.

Common errors to avoid

- Use the graphical and numerical methods to estimate ONLY, but do not blindly reply on what they seem to suggest.

Learning questions for Section D 1-2

Review questions:

1. Describe how the graphical method of estimating limits works.
2. Describe how the numerical method of estimating limits works.
3. Discuss the strengths and weaknesses of the graphical and numerical methods to estimate limits.

Memory questions:

1. What is the main concrete tool used in the graphing and numerical methods to estimate limits?
2. What is the main intellectual tool used in the numerical method to estimate limits?
3. What is the main intellectual tool used in the graphical method to estimate limits?

Computation questions:

For each of the functions provided in questions 1-38:

a) Identify the values for which the limit is worthwhile computing.
b) Use the graphical method to estimate the corresponding limits.
c) Use the numerical method to estimate the corresponding limits.

1. $y=\dfrac{x^3-1}{x^2-1}$

2. $y=\dfrac{x^3+8}{x+2}$

3. $y=\dfrac{5x^2}{3x^2-x}$

4. $y=\dfrac{x+2x^2}{x}$

5. $y=\dfrac{1}{x-2}-\dfrac{4}{x^2-4}$

6. $y=\dfrac{4}{x}-\dfrac{3}{x^2-x}$

7. $y=\dfrac{1}{\sqrt{4-x^2}}$

8. $y=\dfrac{3x}{\sqrt{1-4x^2}}$

9. $y=\dfrac{\sqrt{4-z}-\sqrt{2}}{z^2-4}$

10. $y=\dfrac{\sqrt{x-3}-\sqrt{3}}{x-6}$

11. $y=\dfrac{x-3}{\sqrt{x^2-6x+9}}$

12. $y=\dfrac{2x}{\sqrt{x^2-6x}}$

13. $y=\dfrac{x}{e^x-1}$

14. $y=\dfrac{e^x-1}{x}$

15. $y=\dfrac{e^x-e}{x-1}$

16. $y=\dfrac{e^x-e^2}{x-2}$

17. $y=xe^{1/x}$

18. $y=e^{3/x}$

19. $y=\dfrac{\ln\left(x^2-1\right)}{x^2-4}$

20. $y=\ln\dfrac{2}{x+2}$

21. $y=\left(1+x\right)^{1/x}$

22. $y=\left(1+\dfrac{2}{x}\right)^{1/x}$

23. $y=\dfrac{\cos x-1}{x}$

24. $y=\dfrac{x\sin x}{1-\cos x}$

25. $y=\dfrac{\cos 2x-1}{x-\dfrac{\pi}{2}}$

26. $y=\dfrac{\cos\left(\dfrac{x}{2}\right)}{\pi x-x^2}$

27. $y=\dfrac{\sin x-\sin 2x}{x}$

28. $y=\dfrac{\sin 2x-\sin 4x}{2x}$

29. $y=\dfrac{\sin x}{\sqrt{x}}$

30. $y=\dfrac{\cos x-1}{\sqrt{x}}$

31. $y=\dfrac{\sin x}{\ln x}$

32. $y=\dfrac{\tan 7x}{\sin 3x}$

33. $y=\dfrac{\sin \pi x}{x-1}$

34. $y=\dfrac{x}{|3x+1|-|3x-1|}$

35. $y=\dfrac{|4-x^2|}{x-2}$

36. $y=\sqrt[3]{(x+7)^2}$

37. $y=\begin{cases}\dfrac{3x-5}{x^2-1} & \text{if } x<0\\ \tan x & \text{if } 0\le x\le 2\\ \dfrac{x^2-2x-3}{x^2-3x} & \text{if } x>2\end{cases}$

38. $y=\begin{cases}\dfrac{2x^3+x^2+1}{x^2-1} & \text{if } x<1\\ \sqrt[3]{(x+7)^2} & \text{if } 1\le x\le 2\\ \dfrac{x^2-1}{x^2-9} & \text{if } x>2\end{cases}$

39. Use the graphical method to estimate the value of $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+cx}-1}{x}$ in terms of the parameter *c*.

40. Determine the value of $\lim\limits_{x\to \pi}\dfrac{\tan(\tan x)}{x-\pi}$ by using the numerical method.

Theory questions:

1. Do the numerical and graphical methods provide exact or estimated values?
2. Do the numerical and graphical methods always determine if a limit exists?
3. If I know that $\lim\limits_{x\to 0}\dfrac{x^3+1}{x+1}=1$, what can I say about $\lim\limits_{t\to 0}\dfrac{t^3+1}{t+1}$?

Proof questions:

1. Use the graphical method and you calculator to estimate the value of $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ and $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x$. Then take the cube root of the first limit, the fourth root of the second and use them to make a conjecture about the value of $\lim_{x\to\infty}\left(1+\frac{n}{x}\right)^x$. Briefly describe your method as you go.

2. Both graphical and numerical methods provide only an estimate of the limit we are after; this estimate may be close or may be very incorrect. Here is a nice illustration of what can go wrong with it. We shall try to estimate $\lim_{x\to 0}\sin\left(\frac{\pi}{x}\right)$.
   a) Use the graphical method to estimate this limit: what do you conclude?
   b) Use the numerical method by completing each of the following tables, each of which has $x\to 0^+$. What do you conclude?

x	*f(x)*
0.1	
0.01	
0.001	
0.0001	
0.00001	
Estimate:	

x	*f(x)*
0.3	
0.03	
0.003	
0.0003	
0.00003	
Estimate:	

x	*f(x)*
0.6	
0.06	
0.006	
0.0006	
0.00006	
Estimate:	

x	*f(x)*
0.7	
0.07	
0.007	
0.0007	
0.00007	
Estimate:	

x	*f(x)*
0.8	
0.08	
0.008	
0.0008	
0.00008	
Estimate:	

x	*f(x)*
0.9	
0.09	
0.009	
0.0009	
0.00009	
Estimate:	

Application questions:

1. Use the numerical method to estimate the instantaneous velocity of an object traveling along the *y* axis according to the position function $y = 3e^{-t}$ at time t=2.

Templated questions:

1. Use the graphical and numerical methods to estimate any limit of your choice.

What questions do you have for your instructor?