Chapter 1: Limits and continuity *Section 3*

Basic properties of limits

What you need to know already:	*What you can learn here:*
➤ The basic concepts, notation and terminology related to limits.	➤ Some basic properties that will allow the computation of some simple limits now and more complex ones later.

Evaluating a function at $x = c$ and computing the limit there are not the same thing conceptually, but they turn out to coincide in most situations, especially when we are dealing with what I call *boring* limits. This is a consequence of a fact that I will first state here in a very informal way, then present, without proof, in a list of basic properties.

Knot on your finger:

The "anti-Murphy's Law" of limits

Unless a special feature of a function complicates matters at a given value, a limit may be computed by analyzing each piece of the function separately.

In other words, if there is no indication that something may go wrong with a function at a value, nothing will go wrong.

I call this the anti-Murphy's law to relate it to the more common Murphy's law: "*If anything can go wrong, it will*". For limits, as long as there are no complicating factors, the limit can be computed by breaking the function down into its components and computing them or thinking through them. But be careful!

Warning bells

Sometimes the feature that makes a limit worth computing, and hence problematic, ***may not be obvious or clearly visible***, especially if you are not familiar with the function in question.

Students often have certain algebra misconceptions that lead them to errors. In such cases the real Murphy's Law may actually come into effect, with disastrous consequences on the accuracy of your mathematical work.

Also, I am probably the only one that calls this property the anti-Murphy's law, so do not expect to find this expression in the literature, and do not expect others to know what it means! But for my students it seems to work as a way to convey the concept. So, use it if it helps you, but do not feel obliged to memorize the idea.

In a more formal way, here are some of the basic, practical implementations of this concept. You will see some more among the *Learning questions*.

Technical facts: Limit laws

If $\lim_{x\to c} f(x)=a$ and $\lim_{x\to c} g(x)=b$, with both a and b real numbers, then:

- $\lim_{x\to c}\left(f(x)+g(x)\right)=$
  $=\lim_{x\to c} f(x)+\lim_{x\to c} g(x)=a+b$
- $\lim_{x\to c}\left(f(x)\times g(x)\right)=$
  $=\lim_{x\to c} f(x)\times\lim_{x\to c} g(x)=ab$
- $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}=\frac{a}{b}$ as long as $b\neq 0$
- $\lim_{x\to c} f(x)^{g(x)}=\lim_{x\to c} f(x)^{\lim_{x\to c} g(x)}=a^b$ as long as a and b are not both 0.

Remember that these facts can be a double-edged sword: they can make many calculations easy, but they can also lull you into a false sense of security and the wrong conviction that any simplifying operation can be done on limits. We'll spend some time in later sections analyzing the subtle situations were things can go wrong. For now, watch out!

But let us start by looking at a situation when the laws simplify our life.

Example: $\lim_{x\to 3}\frac{(x-1)^{2x}}{\cos\pi x}$

This may seem like a complicated limit, but when we look at the function, we realize that there is nothing wrong with it at or around $x=3$. So, we break it down into its parts:

$$\lim_{x\to 3}\frac{(x-1)^{2x}}{\cos\pi x}=\frac{\lim_{x\to 3}(x-1)^{\lim_{x\to 3} 2x}}{\lim_{x\to 3}\cos\pi x}=\frac{2^6}{\cos 3\pi}=-64$$

There is another important aspect of limits that is linked to a situation where a limit does not exist, but we can say something more and useful about why.

Definition

If $f(x)$ can become arbitrarily large as x approach c, we say that the function ***approaches infinity***, or that its limit at c is ***infinite*** and write:

- $\lim_{x\to c} f(x)=\infty$ if the values of $f(x)$ become ***ever larger and positive*** as $x\to c$.
- $\lim_{x\to c} f(x)=-\infty$ if the values of $f(x)$ become ***ever larger and negative*** as $x\to c$.

The same applies to one-sided limits.

There are two important issues to keep in mind about infinite limits.

Warning bells

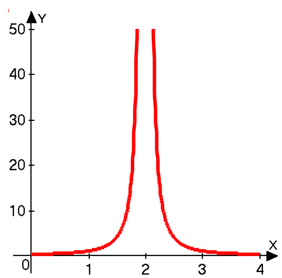
From the technical point of view, if a limit is ***infinite***, it ***does not exist***.

The notation $\lim_{x \to c} f(x) = \pm\infty$ is meant to provide ***useful information*** about the behaviour of the function, ***NOT*** to assert the existence of a limit.

Warning bells

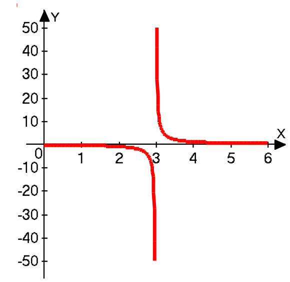
The symbols ∞ and -∞ represent ***concepts*** and ***not numbers***. They represent an interesting ***limiting behaviour*** of a function. Therefore, usual algebraic rules do NOT apply to them and we cannot use the anti-Murphy's law when it involves infinite limits.

Example: $\lim_{x \to 2} \frac{1}{(x-2)^2}$



This is a fraction whose denominator approaches 0, so we cannot use the anti-Murphy's law, as we cannot divide by 0. However we notice that by letting *x* approach 2, this denominator can be made arbitrarily small – and positive – so that the whole function can be made arbitrarily large and positive, as the graph shows. We can indicate this by writing: $\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty$

Example: $\lim_{x \to 3} \frac{1}{x-3}$



This time the denominator again approaches 0, but it is positive to the right of 3 and negative to the left of 3. So we can use the same logical process on each side separately:

$$\lim_{x \to 3^-} \frac{1}{x-3} = -\infty \quad ; \quad \lim_{x \to 3^+} \frac{1}{x-3} = \infty$$

However, notice that this time the two limits provide two different types of infinity. Therefore we cannot say that the two-sided limit is infinite: which infinity would it be? Instead we conclude that such limit does not exist:

$$\lim_{x \to 3} \frac{1}{x-3} = DNE$$

Again, the calculator's graph confirms this conclusion.

The last two examples are instances of another basic and very useful property of limits, one to which I have also given a special, but not commonly used name.

Technical fact:

The limit "law of balloons"

If $\lim_{x \to c} f(x) = a \neq 0$ and $\lim_{x \to c} g(x) = 0$, then

- $\lim_{x \to c} \frac{f(x)}{g(x)} = \infty$ if $\frac{a}{g(x)} > 0$ as $x \to c$.
- $\lim_{x \to c} \frac{f(x)}{g(x)} = -\infty$ if $\frac{a}{g(x)} < 0$ as $x \to c$.

This situation is usually referred to as a ***#/0 form*** and applies to one-sided limits as well.

So, why do you call it the law of balloons?

Because it evokes the image of balloons in the sense that as $x \to c$, the denominator becomes very small, hence very light, thus pushing the whole fraction up to infinity! It is an image that appeals to me and to some students, but not to everyone. Use it only if it helps you.

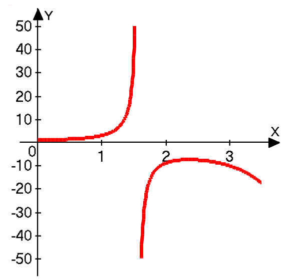
Example: $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cosh x}{\cos x}$

This looks like a complex limit, but again, we can think our way through it. The hyperbolic cosine function always generates positive values, so, whatever $\cosh \frac{\pi}{2}$ is, it is positive. On the other hand $\cos \frac{\pi}{2} = 0$, but it approaches 0 from different sides.

Therefore, we need to look at the one-sided limits separately:

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\cosh x}{\cos x} = \frac{+\#}{+0} = \infty$$

$$\lim_{x \to \frac{\pi}{2}^+} \frac{\cosh x}{\cos x} = \frac{+\#}{-0} = -\infty$$



I assume and hope that the notation I just used to indicate whether we are dealing with positive or negative numbers is clear ☺.

The graph confirms this conclusion.

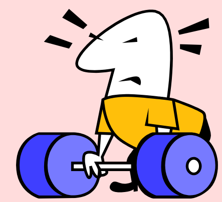
There is another useful and practical property that has earned a special name from me.

Technical fact:

The limit "law of gravity"

If $\lim\limits_{x \to c} f(x) = h$ and $\lim\limits_{x \to c} g(x) = \pm\infty$ (meaning that both options are acceptable), then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = 0$$

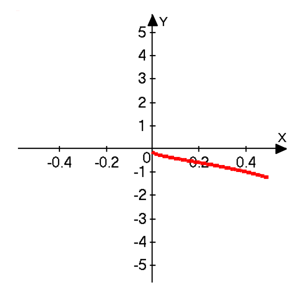


This is usually referred to as a ***#/∞ form*** and applies to one-sided limits as well. I use the image of gravity because here the denominator is becoming large – think of it as heavy – thus pushing the function into the ground, at level 0. As always, use the image at your discretion.

Example: $\lim\limits_{x \to 0^+} \dfrac{\cos x}{\ln x}$

We know from the properties of these functions, that the numerator approaches 1 and the denominator approaches -∞. Therefore we can easily conclude that this limit is 0.

And once again, the graph supports the conclusion, even though the graph itself is not totally convincing. But this time we have logic on our side.



Finally, there is another important property that has a strange name, but this time it is a name given by someone else and recognized by the whole mathematical community.

Technical fact:

The "Squeeze Theorem"

Assume that $c \in (a,b)$, meaning that $a < c < b$, and that:

$$f_1(x) \le g(x) \le f_2(x)$$

for every $x \in (a,b)$ except possibly for $x = c$. If

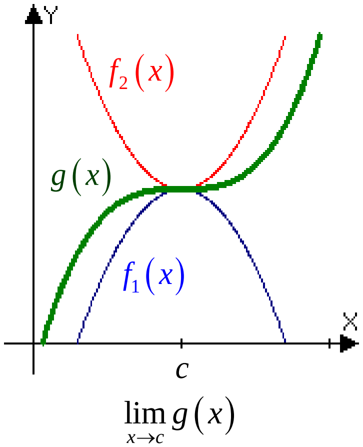
$$\lim_{x\to c} f_1(x) = L \quad and \quad \lim_{x\to c} f_2(x) = L$$

then:

$$\lim_{x\to c} g(x) = L$$

Although this theorem and its name are traditional items in a calculus course, it is mainly a technical theorem, with few practical applications of note. This is because usually proving the inequality among the three required functions is more difficult than computing the limit itself with other methods!

We shall not see or use this theorem much, if at all, so I will not give you any examples yet. The formal proof of this theorem is also rather technical, requiring the formal, technical definition of a limit, but its content is very intuitive and demonstrated by this picture:



I trust that the picture well illustrate the reason why the name of "*Squeeze Theorem*" is used. And you may also see it referred to as the "*Sandwich Theorem*", especially by people like me who enjoy food!

Summary

- Common sense principles can be used to determine a limit when the quantities involved can be related in a definite way.
- Limits of this kind can be computed formally, but using logic can provide a faster and equally safe method. But still use with care!

Common errors to avoid

- Be careful in the use of the three laws presented in this section and avoid hidden issues that may make them invalid. Examples of such issues will be seen later.

Learning questions for Section D 1-3

Review questions:

1. Describe in your own words, but accurately, the concept I call "*Anti-Murphy's Law*".
2. Explain what it means for a function to have an infinite limit.
3. Describe in your own words, but accurately, the concept I call "*Law of balloons*".
4. Describe in your own words, but accurately, the concept I call "*Law of gravity*".
5. Describe in your own words, but accurately, the "*Squeeze theorem*".
6. Explain why if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \pm\infty$, depending on the sign.
7. Explain why if $f(x)$ is bounded as $x \to c$ and $\lim_{x\to c} g(x) = \pm\infty$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = 0$.

Memory questions:

1. What is the value of $\lim_{x\to 0} \frac{1}{x}$?
2. What is the value of $\lim_{x\to 0^-} \frac{1}{x}$?
3. What is the value of $\lim_{x\to 0^+} \frac{1}{x}$?
4. What is the value of $\lim_{x\to 0^+} \ln x$?
5. When is it true that $\lim_{x\to a}\left(f(x) + g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$?
6. When is it true that $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$?
7. When is it true that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$?

Computation questions:

Use any of the laws presented in this section to compute the limits presented in questions 1-34.

1. $\lim_{x\to -1} \frac{x^3-1}{x^2-1}$

2. $\lim_{x\to 2} \frac{x^3+8}{x+2}$

3. $\lim_{x\to \frac{1}{3}^+} \frac{5x^2}{3x^2-x}$

4. $\lim_{x\to 1} \frac{x+2x^2}{x}$

5. $\lim_{x\to -2} \left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$

6. $\lim_{x\to 1} \left(\frac{4}{x}-\frac{3}{x^2-x}\right)$

7. $\lim_{x\to 2^-} \frac{1}{\sqrt{4-x^2}}$

8. $\lim_{x\to \frac{1}{2}^-} \frac{3x}{\sqrt{1-4x^2}}$

9. $\lim_{z\to -2} \frac{\sqrt{4-z}-\sqrt{2}}{z^2-4}$

10. $\lim_{x\to 7} \frac{\sqrt{x-3}-\sqrt{3}}{x-6}$

11. $\lim_{x\to 3} \frac{x-3}{\sqrt{x^2-6x+9}}$

12. $\lim_{x\to 0} \frac{x-3}{\sqrt{x^2-6x+9}}$

13. $\lim_{x\to 0^-} \frac{2x}{\sqrt{x^2-6x}}$

14. $\lim_{x\to 6^+} \frac{2x}{\sqrt{x^2-6x}}$

15. $\lim_{x\to 0} \frac{e^x-1}{x-1}$

16. $\lim_{x\to 0} \frac{x-1}{e^x-1}$

17. $\lim_{x\to 1} e^{3/x}$

18. $\lim_{x\to 0^-} xe^{1/x}$

19. $\lim_{x\to 0} \frac{\ln\left(x^2-1\right)}{x^2-4}$

20. $\lim_{x\to 0} \ln \frac{2}{x+2}$

21. $\lim_{x\to 1} (1+x)^{1/x}$

22. $\lim_{x\to 1} (1+x)^x$

23. $\lim_{x\to 0} \frac{x+\cos x}{1-\cos x}$

24. $\lim_{x\to \pi} \frac{x\sin x}{1-\cos x}$

25. $\lim_{x\to -\pi} \frac{\cos x-1}{x}$

26. $\lim_{x\to \frac{\pi}{2}^+} \frac{\cos 2x-1}{x-\frac{\pi}{2}}$

27. $\lim_{x\to \infty} \frac{\sin x-\sin 2x}{x}$

28. $\lim_{x\to \frac{\pi}{2}} \frac{\sin x-\sin 2x}{x}$

29. $\lim_{x\to 0^+} \sqrt{x}\sin x$

30. $\lim_{x\to\pi}\frac{\cos x-1}{\sqrt{x}}$

31. $\lim_{x\to 1^-}\frac{\sin x}{\ln x}$

32. $\lim_{x\to\frac{\pi}{2}^-}\frac{\tan 7x}{\sin 3x}$

33. $\lim_{x\to 0}\frac{x}{|3x+1|-|3x-1|}$

34. $\lim_{x\to 2^+}\frac{|4-x^2|}{x-2}$

Theory questions:

1. Is it true that if k is a real number and $\lim_{x\to a} f(x)=h$, then $\lim_{x\to a} kf(x)=kh$?

Proof questions:

1. Determine whether $\lim_{x\to 1}\sin(x-1)\cos\left(\frac{3}{x-1}\right)$ exists or not. If it does, find it.

Templated questions:

1. Try applying the methods discussed in this section to any limit you may need or see.

What questions do you have for your instructor?