Chapter 1: Limits and continuity **Section 4**

Vertical asymptotes

What you need to know already:	What you can learn here:
➤ The concept and basic properties of limits.	➤ The graphical interpretation of infinite limits as ***vertical asymptotes***..

We have seen that an interesting situation arises when, in a limit computation, the function, or some part of it, becomes arbitrarily large, without bounds. Let me review the concept.

Definition

The notation $\lim_{x \to c} f(x) = \pm\infty$ means that, as $x \to c$, $f(x)$ can be made as large as we want, in positive or negative values respectively, by selecting *x* sufficiently close to *c*.

A limit of the form $\lim_{x \to c} f(x) = \pm\infty$ is called an ***infinite limit***.

The graphical interpretation of such a limit is that as *x* approaches *c*, the graph of the function either goes up indefinitely, or down indefinitely. This leads to a commonly used terminology.

Definition

If $x = c$ is a ***finite*** value and any ***one*** of the following limit conditions occur, we say that $y = f(x)$ has a ***vertical asymptote*** at $x = c$:

- $\lim_{x \to c^-} f(x) = \infty$
- $\lim_{x \to c^-} f(x) = -\infty$
- $\lim_{x \to c^+} f(x) = \infty$
- $\lim_{x \to c^+} f(x) = -\infty$

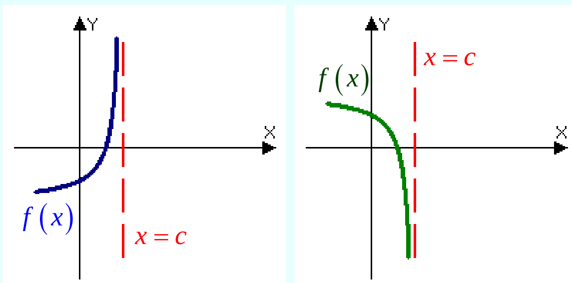
Notice that the first two conditions cannot occur at the same time and neither can the last two conditions. However, either one of the first two conditions can occur together with either one of the last two and vice versa.

What?

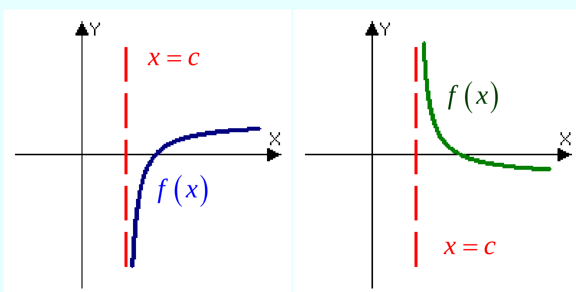
What I am saying is that the left and right sides are separate. The following definitions and observation will probably clarify the issue.

Definition

- If $\lim_{x\to c^-} f(x)=\infty$ or $\lim_{x\to c^-} f(x)=-\infty$, $f(x)$ has a ***right*** vertical asymptote at $x=c$:

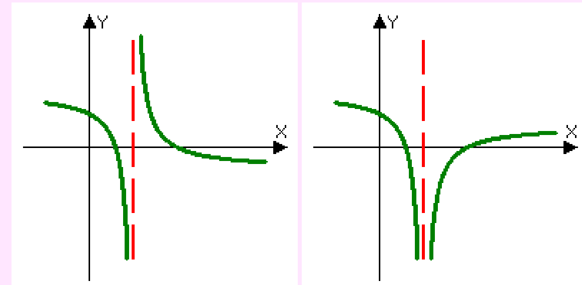


- If $\lim_{x\to c^+} f(x)=\infty$ or $\lim_{x\to c^+} f(x)=-\infty$, $f(x)$ has a ***left*** vertical asymptote at $x=c$:



Warning bells

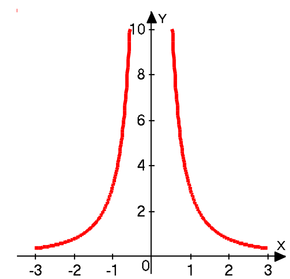
The line $x=c$ may be both a left and right vertical asymptote, and it may be so with different types of infinities on either side:



Although you may have seen vertical asymptotes before, here are some basic examples. Pay attention to what is used to determine the presence of a vertical asymptote: is it the same criterion you have used so far?

Example: $y=\dfrac{3}{x^2}$

Here we notice that we can let *y* become as large as we want by letting *x* become sufficiently close to 0. The closer *x* is to 0, the larger $f(x)$ becomes. This is the rationale behind the "*law of balloons*" that we saw in the last section.

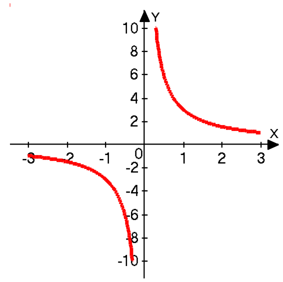


Therefore, $x=0$ is a vertical asymptote, both on the left and on the right and on both sides we have $\lim_{x\to 0}\dfrac{3}{x^2}=\infty$.

Example: $y = \dfrac{3}{x}$

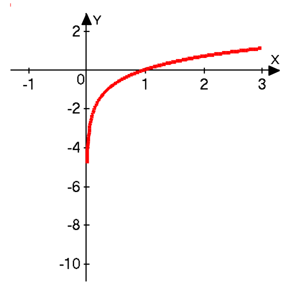
Here we also notice that we can let *y* become as large as we want by choosing *x* sufficiently close to 0, so that $x = 0$ is a vertical asymptote, But this time the function is approaching a different type of infinity from each of the two sides, a fact reflected in the graph:

$$\lim_{x \to 0^-} \frac{3}{x} = -\infty \quad ; \quad \lim_{x \to 0^+} \frac{3}{x} = \infty .$$



Example: $y = \ln x$

We know that the natural logarithm is only defined for positive values of *x* and that it becomes increasingly large in negative values as *x* approaches 0. Therefore $\lim_{x \to 0^+} \ln x = -\infty$ and $x = 0$ is a left vertical asymptote.



Notice that this function approaches infinity so fast that the computer program is only able to draw a piece of the curve. That does not mean that the curve stops at some value close to 0, only that the computer cannot calculate how far down it is, or that it is too close to the vertical axis to be seen.

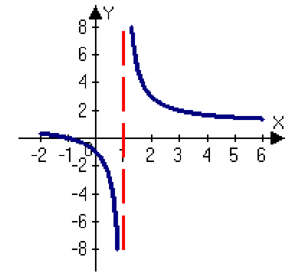
Do vertical asymptotes always occur at x=0?

No, but we math teachers are afraid of going too far, so we always try to stay close to 0 ☺. Here are two basic and somewhat familiar examples of different locations, and we'll see many more in future work.

Example: $y = \dfrac{x+1}{x-1}$

At $x = 1$ we end up with a #/0 situation and our law of balloons tells us that:

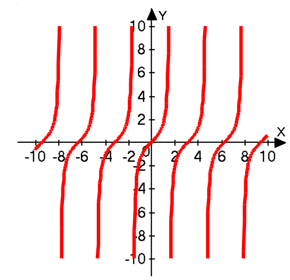
$$\lim_{x \to 1} \frac{x+1}{x-1} = \pm\infty$$



By checking further, we see that it goes to positive infinity to the right of the asymptote and to negative infinity to the left.

Example: $y = \tan x$

Since $\tan x = \dfrac{\sin x}{\cos x}$, whenever $\cos x = 0$ the function will have a vertical asymptote, since at those values $\sin x \neq 0$. But this happens for any value of the form $x = \dfrac{\pi}{2} + k\pi$, that is, infinitely many times. That is fine: a function can have any number of vertical asymptotes and even infinitely many, as is the case here for the lines $x = \dfrac{\pi}{2} + k\pi$.



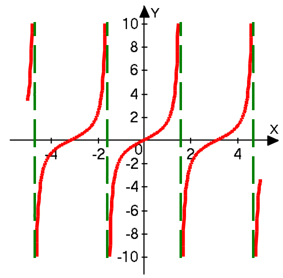
Why didn't you draw the asymptotes in the last example?

In some of the examples the asymptote was the *y* axis, so it was difficult to make it visible. But there is another important reason for not drawing them that is often ignored by students, but should not be.

Warning bells

A vertical ***asymptote is NOT part of the graph*** of a function, so that when it is useful to show it, it should be drawn in a way that clearly distinguishes it from the function itself.

So, if you really want to see the asymptotes of the tangent function, you should draw them as dashed lines and perhaps even with a different colour, as shown here.



Notice that a graphing calculator will often show a vertical asymptote as if it were part of the graph. This is a glitch due to the way the calculator draws the graph. It can be avoided by choosing a window that has the vertical asymptote exactly in the middle of the window.

It is now time to bring up a very important point.

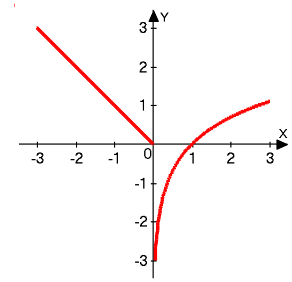
Warning bells

Despite what you may have learned in your school introduction to asymptotes, their definition has ***nothing to do with intersections***, but only on the existence of an ***infinite limit***.

Although in most cases that we shall see a function will not have a point in common with its vertical asymptotes, this is ***NOT a requirement***!

Example: $y=\begin{cases}-x & \text{if } x\le 0\\ \ln x & \text{if } x>0\end{cases}$

This is a piecewise function, which does include a point at $x=0$, namely the origin. However, since $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\ln x=-\infty$, the line $x=0$ is also a vertical asymptote for this function.



This is an example of a function that has an asymptote and touches it as well!

This last example is a bit contrived, but it is a valid one that emphasizes the fact that vertical asymptotes are determined by the presence of infinite limits, not by lack of intersections. Keep this distinction clearly in mind and reflect on it.

Strategy for discovering vertical asymptotes

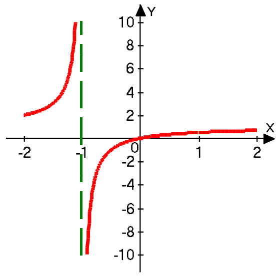
Vertical asymptotes occur mostly in one of these situations:

- When the function includes a ***fraction*** whose denominator becomes 0 for some value of *x*.
- When the function includes a ***transcendental*** portion that is ***known to have*** a vertical asymptote.

However, in both cases the existence of the vertical asymptote must be checked by computing the relevant limit, since other graphical features may also correspond to such situations.

Example: $y = \dfrac{x^2 - x}{x^2 - 1}$

In this case there is a fraction with a denominator that becomes 0 at $x = \pm 1$. But this is not sufficient to guarantee the existence of vertical asymptotes there: we need to check the limit.

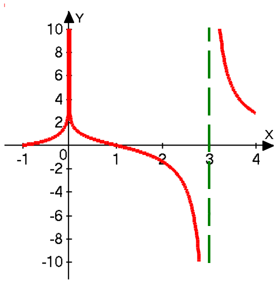


For $x = -1$ we see that the top approaches 2 and the bottom approaches 0. By the law of balloons, the function approaches infinity there and we do get a vertical asymptote at $x = -1$.

However, for $x = 1$ both top and bottom become 0 and we cannot use the law of balloons. We need more methods to analyze this case, but a calculator graph suggests that there is no asymptote there.

Example: $y = \dfrac{\ln x^2}{x - 3}$

This function includes a denominator, which becomes 0 for $x = 3$, and the logarithm function, which has a vertical asymptote at $x = 0$. If $x = 3$, the law of balloons guarantees an asymptote, while at $x = 0$ we end up with an unbounded quantity divided by a number close to -3. In both cases we get a vertical asymptote, as shown in the graph.



Example: $y = \dfrac{\sin x}{x}$

The fact that the denominator becomes 0 at $x = 0$ tempts us to conclude that there is an asymptote there. But the law of balloons does not apply there! In fact, a calculator graph confirms that no asymptote seems to be there. But we need more methods to confirm that.

Summary

- A vertical asymptote occurs whenever a limit at a finite value is infinite.
- Vertical asymptotes have to do with infinite limits, not with the absence of intersections.

Common errors to avoid

- Do not require vertical asymptotes to have more properties than they need to have. An infinite limit on either side is the only requirement.

Learning questions for Section D 1-4

Review questions:

1. Describe what a vertical asymptote is.

2. Discuss the relationship among vertical asymptotes, points of intersection and limits.

3. Explain how we can detect the possible presence of a vertical asymptote and how we can verify it.

Memory questions:

1. Which limit conditions identify a vertical asymptote?

2. If $\lim_{x \to c^-} f(x) = \infty$, what is the equation of the corresponding asymptote?

3. How many vertical asymptotes can a function have?

4. Which is the vertical asymptote of the natural logarithmic function?

5. Where do the vertical asymptotes of the tangent function occur?

Computation questions:

For each of the functions presented in questions 1-30:

a) Identify the values of x where a vertical asymptote may possibly be present.
b) Attempt to verify its presence by computing the relevant limit, preferably by using one of the limit properties we have seen so far, or, if that is not possible, by using a calculator's graph.

1. $y = \dfrac{x^3 - 1}{x^2 - 1}$

2. $y = \dfrac{x^3 - 8}{x + 2}$

3. $y = \dfrac{5x^2}{3x^2 - x}$

4. $y=\frac{x+2x^2}{x}$

5. $y=\frac{1}{x-2}-\frac{4}{x^2-4}$

6. $y=\frac{4}{x}-\frac{3}{x^2-x}$

7. $y=\frac{x^3}{x-1}$

8. $y=\frac{x^2+x+12}{x-3}$

9. $y=\frac{1}{\sqrt{4-x^2}}$

10. $y=\frac{3x}{\sqrt{1-4x^2}}$

11. $y=\frac{\sqrt{4-x}-\sqrt{2}}{x^2-4}$

12. $y=\frac{\sqrt{x-3}-\sqrt{3}}{x-6}$

13. $y=\frac{\sqrt{x+3}}{x^2-25}$

14. $y=\frac{x-3}{\sqrt{x^2-6x+9}}$

15. $y=\frac{x}{e^x-1}$

16. $y=\frac{e^x-1}{x}$

17. $y=e^{3/x}$

18. $y=e^x x^{-2}$

19. $y=\frac{x+4}{\left(e^x-1\right)^2}$

20. $y=\frac{\ln\left(x^2-1\right)}{x^2-4}$

21. $y=\ln\frac{2}{x+2}$

22. $y=(1+x)^{1/x}$

23. $y=\left(1+\frac{2}{x}\right)^{1/x}$

24. $y=\frac{x\sin x}{1-\cos x}$

25. $y=\frac{\cos(x)-1}{x}$

26. $y=\frac{\tan(7x)}{\sin(3x)}$

27. $y=\frac{\sin(2x)}{\sinh x}$

28. $y=\frac{\cos(2x)-1}{x-\frac{\pi}{2}}$

29. $y=\frac{\sin x}{\sqrt{x}}$

30. $f(x)=\begin{cases} e^x & \text{if } x\le 0 \\ \sin x & \text{if } x>0 \end{cases}$

31. $y=\begin{cases} 3 & \text{if } x\le 0 \\ 1/x & \text{if } x>0 \end{cases}$

32. $y=\begin{cases} e^x & \text{if } x\le 2 \\ \frac{x^3-8}{x^2-4} & \text{if } x>2 \end{cases}$

33. $y=\begin{cases} \frac{3x-5}{x^2-1} & \text{if } x<0 \\ \tan x & \text{if } 0\le x\le 2 \\ \frac{x^2-2x-3}{x^2-3x} & \text{if } x>2 \end{cases}$

34. $y = \begin{cases} \dfrac{2x^3 + x^2 + 1}{x^2 - 1} & \text{if } x < 1 \\ \sqrt[3]{(x+7)^2} & \text{if } 1 \le x \le 2 \\ \dfrac{x^2 - 1}{x^2 - 9} & \text{if } x > 2 \end{cases}$

Theory questions:

1. Which line is a vertical asymptote for $y = \ln(1 - x)$?

2. How can you force your graphing calculator to NOT show a vertical asymptote?

3. Can the same function have a *y* intercept as well as the *y* axis as a vertical asymptote?
   Yes

Templated questions:

1. Identify any vertical asymptotes for any function on which you are working.

What questions do you have for your instructor?