Chapter 1: Limits and continuity — **Section 6**

Continuity

What you need to know already:	What you can learn here:
➤ The concept of limit for a function.	➤ What it means for a function to be ***continuous***, both at a value and overall.

When I first introduced the concept of a limit, I emphasized the fact that, although it is technically possible to compute the limit of a function at any value, it only makes sense to do so at those special values for which something interesting or unusual occurs in the function. At values where nothing much is happening I say that the limit is *boring*. Well, the concept of continuity is linked to this idea.

While continuity is an intuitively simple concept, it proved very complex and counterintuitive when mathematicians tried to study it rigorously. Since it plays a very important technical role in calculus, it is important to reflect on it, but here I will do so informally, without dwelling on its technical difficulties.

Here is the intuitive idea, the one we are trying to achieve.

> **Definition**
>
> A function $y = f(x)$ is ***continuous at a value*** $x = c$ if its graph has ***no breaks*** of any kind at that value.
>
> A function $y = f(x)$ is ***discontinuous***, or has a ***discontinuity*** at $x = c$, if its graph has some type of break at that value.

You will often hear or read that a function is simply *continuous*, without specifying where. When I say that, I mean the following.

> **Definition**
>
> A function is ***continuous*** if it has no discontinuities between any two values at which it is continuous.

However, for a number of good reasons, the following definition is also used, especially in technical settings:

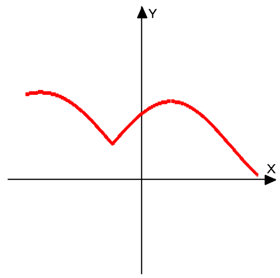
A function $y = f(x)$ is continuous if it is continuous at all values in its domain.

The problem with this definition is that it considers as continuous many functions whose graph have clear breaks – such as vertical asymptotes – but at values outside the domain.

I prefer to highlight the feature provided by the breaks, hence the version I use.

Most functions that we shall consider are continuous at most and even all values. For instance, all polynomial and exponential functions are continuous at every value and rational functions only have a few vertical asymptotes.

Unfortunately, it is difficult, if not impossible, to decide whether a function is continuous by simply looking at its graph. For instance, can you see the two discontinuities of the function whose graph is shown here? No? Well, that's my point!



So, how do we know whether a seemingly continuous graph contains some invisible discontinuity? We need a more formal definition of what we mean by *continuity*, one that can be verified by appropriate computations.

Definition

A function $y = f(x)$ is said to be ***continuous at a value*** $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

This requirement is equivalent to the following three conditions, taken together:

1. $f(c)$ ***exists***, that is, c is in the domain of $f(x)$.
2. $\lim_{x\to c} f(x)$ ***exists***, as a unique, finite number.
3. the two above values are ***equal***.

If one of the above three conditions is false, then the function is said to be ***discontinuous*** at $x = c$.

Example:

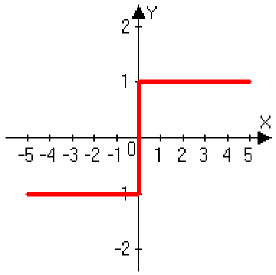
I just claimed that the function whose graph is shown above has two discontinuities, even though the graph seems continuous. That is because its formula is:

$$y = \begin{cases} \dfrac{\sin(x-1)}{1-x} + 0.9 & if\ \ x < -1 \\ \dfrac{\sin(x-1)}{x-1} & if\ \ x \geq -1 \end{cases}$$

From this formula you can see that the function is not defined – and hence discontinuous – at $x = 1$, since the denominator becomes 0 there. Also, if you compute the left and right limits at $x = -1$, you will notice that they are not equal, so that the limit itself does not exist: another discontinuity.

The formal definition allows us to identify and check the presence of discontinuities that could not be clearly seen in the graph.

Example: $f(x) = \dfrac{|x|}{x}$



This is the so-called "*signum*" function, which assign to each real number the value +1 if the number is positive and -1 if it is negative. What happens at 0? The computer graph shown here seems to suggest a sharp climb from -1 to 1, but that cannot be the case, since the function can only generate ± 1, but nothing in between!

The reality is that at $x = 0$ the function is not even defined, thus discontinuous. We'll see later how to describe and classify this kind of discontinuity.

So, why does the computer get it wrong?

Because the computer does not graph every point of the curve, but only certain ones dictated by the chosen window and screen resolution. In the case of the signum function, the value $x = 0$ was not used, so the computer did not see the discontinuity and tried to get around it with a sharp rise.

Nice attempt by the computer to fool us, but I now hope that you will not fall for something like this anymore!

And if function has a limited domain, what happens at the boundary values?

Good point. Here are some basic examples.

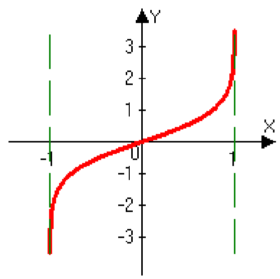
Example: $f(x)=\sqrt{1-x^2}$

This function describes the upper half of the unit circle. Its domain is the interval $[-1,1]$ and it is discontinuous only at the end points, where it stops. Therefore, we can say that this function is continuous, whether we use my preferred definition or the more technical one.

Example: $g(x)=\ln x$

This function is only defined for $x>0$, but it has no other problems within its domain, which does not include $x=0$. Therefore, we can simply say that this function is continuous.

Example: $y=\tanh^{-1}x$



Remember that this is the inverse hyperbolic tangent function, whose graph is shown here, NOT the hyperbolic cotangent, which has a discontinuity at $x=0$, its vertical asymptote.

Since there are no breaks or other troublesome features within its domain of $(-1,1)$, we say that it is continuous, even though at the end points of the domain it is not even defined.

It looks like one-sided limits could play a role here!

Yes, so let's let them!

Definition

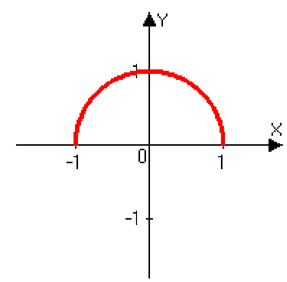
A function $y=f(x)$ is ***left continuous at*** $x=c$ if

$$\lim_{x\to c^-} f(x)=f(c)$$

A function $y=f(x)$ is ***right continuous at*** $x=c$ if

$$\lim_{x\to c^+} f(x)=f(c)$$

Example: $f(x)=\sqrt{1-x^2}$



The upper half of the unit circle is not continuous at its end points, $x=\pm1$, since no limit exists there: the function can only be evaluated on one side for each. But we look for continuity on that side!

$$\lim_{x\to1^-}\sqrt{1-x^2}=0=\sqrt{1-1^2}$$

So, this function is left continuous at $x=1$. Also:

$$\lim_{x\to-1^+}\sqrt{1-x^2}=0=\sqrt{1-(-1)^2}$$

Therefore it is right continuous at $x=-1$.

How are these formal definitions related to the informal one?

By requiring the value of the function and of the limit to exist and be the same, we require that the graph actually reaches the value that it is approaching. This guarantees the absence of a break.

Notice that our *Anti-Murphy's law* tells us that any function whose formula or definition does not include potentially troublesome features is continuous. This includes most familiar functions, such as polynomials, exponentials, sine and cosine and any rational function whose denominator is never 0.

Before we close this section, duty calls me to bring to your attention an important theorem that is used to prove formally many other technical theorems. Its statement is quite clear and reasonable, but its formal proof is very technical and we shall not use it for anything practical. So, here it is, for your general information, with an example of the only use we can make of it at this point.

Technical fact:

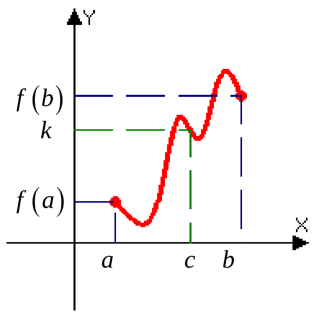
The intermediate value theorem

If $f(x)$ is ***continuous on a closed interval*** $[a,b]$ and $f(a) \le k \le f(b)$, then there is at least one value c such that $a \le c \le b$ and $f(c) = k$.

In more concrete terms, as we trace the graph of such a function starting at $(a, f(a))$ and going all the way to $(b, f(b))$, we shall find at least one point whose y coordinate is between $f(a)$ and $f(b)$.

This is the traditional picture that accompanies this theorem.

Notice that the graph here goes through points for which $y = k$ three times and it also reaches below $f(a)$ and above $f(b)$, but that is not the issue! The theorem simply claims that the curve must go through every height in between, at least once.



So, it says that to go from here to there, you must go through everything in between!

Yes, unless you have a Star Trek transporter, that is, a discontinuity!

And what is the example you mentioned of the only use we can make of this?

Example: $\cos x = x$

Does this equation have a solution? No solution can be found with algebraic methods or by trial and error, but do we even know IF there is one solution?

This question is equivalent to asking whether the equation $\cos x - x = 0$ has a solution and this, in turn, is the same as asking whether the function $f(x) = \cos x - x$ ever becomes zero. But we can see that this function is continuous (no values cause problems) and:

$$f(0) = \cos 0 - 0 = 1 > 0 \quad ; \quad f(\pi) = \cos \pi - \pi = -1 - \pi < 0$$

By the intermediate value theorem, this function, which goes from a positive to a negative number, must go through everything in between, including 0! That means that yes, the equation has a solution.

As for what that solution is, or how to find it, the theorem tells us nothing. And that is why this theorem is useful (an existence theorem), but not very practical.

A more practical and arguably more interesting question is how to identify and describe the discontinuities of a function. That is what we'll start answering in the next section and continue on in the next chapter. At that point we'll start looking at all kinds of interesting situations, both algebraically and graphically.

Summary

- A function is continuous at the values where its graph is NOT broken. This statement captures the essence of the idea, but is not precise enough to allow verification.
- The formal definition of continuity requires the limit of the function to exist at the given value and to equal the value of the function.

Common errors to avoid

- Do not assume that a function is continuous simply because it seems so and you cannot identify its discontinuities. Sometimes a careful analysis is needed to discover them.

Learning questions for Section D 1-6

Review questions:

1. Explain what the concept on continuity means in practical terms.
2. Describe the technical requirements for continuity at a value.

Memory questions:

1. Which single equation defines the concept of continuity?
2. Which three conditions are required for continuity at $x = c$?

Computation questions:

For each of the functions presented in questions 1-16, identify the values of *x* for which the function may be discontinuous. No need to check for continuity: we'll do that after we see more methods for computing limits.

1. $y = \dfrac{x+1}{x^2 - x}$
2. $y = \dfrac{x^2 - 5}{x^3 - 5\sqrt{5}}$
3. $y = \sqrt{2x^3 + 8x} - \sqrt{2x^3}$

4. $y=\dfrac{x-3}{\sqrt{x^2-6x+9}}$

5. $y=\dfrac{\sqrt{3x^2+5x}}{9x+5}$

6. $y=\dfrac{3x}{\sqrt{1-4x^2}}$

7. $y=e^{\frac{1}{x^2-4}}$

8. $y=\dfrac{2}{e^{-x}+5}$

9. $y=\ln\left(\dfrac{x^3+x^2}{x-1}\right)$

10. $y=\ln\left(x^2-x^3\right)$

11. $y=\dfrac{\sec x-1}{x\sec x}$

12. $y=\dfrac{3x}{\sin^{-1}2x}$

13. $y=\begin{cases}2\sqrt{4-x} & if\ x\le 4\\ \dfrac{2}{4+x} & if\ x>4\end{cases}$

14. $y=\begin{cases}\dfrac{3}{x-5} & if\ x\le 2\\ \dfrac{5}{x+3} & if\ x>2\end{cases}$

15. $f(x)=\begin{cases}\dfrac{1}{2-x^2} & if\ x<0\\ \dfrac{e^x}{2-x} & if\ 0\le x\le 1\\ \dfrac{2}{1-x} & if\ x>1\end{cases}$

16. $f(x)=\begin{cases}\cos\pi x & if\ x<\dfrac{1}{3}\\ \sqrt{\dfrac{1}{2x}-1} & if\ x>\dfrac{1}{3}\end{cases}$

17. Which value of *a* makes the function $f(x)=\begin{cases}xe^{\frac{1}{x}} & if\ x\ne 0\\ a+2 & if\ x=0\end{cases}$ left continuous for all values of *x*?

18. Which values of *a* and *b* make the following function continuous?

$$f(x)=\begin{cases}2\ln x^2 & if\ x<-2\\ ax+b & if\ -2\le x<0\\ 8x & if\ x\ge 0\end{cases}$$

19. Use proper limit notation to determine the value of *a* for which the following function is continuous:

$$f(x)=\begin{cases}x+2 & if\ x\le a\\ \log_a a^3x & if\ x>a\end{cases}$$

20. Given the function $f(x)=\begin{cases}x^2\cos\dfrac{1}{x} & if\ x\ne 0\\ 0 & if\ x=0\end{cases}$, use the definition of continuity to show that the function is continuous at *x*=0.

Theory questions:

1. Provide an example of a function that has infinitely many discontinuities.
2. Are there functions that are not continuous at any point in their domain?
3. Is it possible for a function to be left continuous and right continuous, but not continuous?
4. Does the presence of horizontal asymptotes affect the continuity of a function?
5. Is it true that if $f(c)$, $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ all exist, then $f(x)$ is continuous at *c*?
6. Is it true that if $f(x)$ is continuous at *c*, computing $\lim_{x\to c} f(x)$ is not worthwhile?
7. What is the main application of the *Intermediate Value Theorem*?

Proof questions:

1. Show that the equation $x^2 \cos x + e^{\cos x} = 0$ has at least two solutions.
2. Show that the equation $\frac{x^3}{10} + 4 = 3e^{\cos x}$ has at least three solutions.

Templated questions:

1. Identify the potential values at which any function you are using may be discontinuous.

What questions do you have for your instructor?