Chapter 1: Limits and continuity **Section 7**

Discontinuities

What you need to know already:	What you can learn here:
➤ The concept and definition of continuity.	➤ The different types of discontinuity and how to recognize them.

When we are dealing with a *boring* limit, we know that we have continuity. But what about the interesting limit situations? How can we analyze them to discover if we are really dealing with a discontinuity and, if so, of what type?

The tool to use for this purpose is the key equality that defines continuity:

$$\lim_{x\to c} f(x) = f(c).$$

Strategy for identifying and classifying discontinuities

To identify and classify the discontinuities of a function $f(x)$:

1. ***Identify*** the values for which something ***unusual*** occurs in the formula or definition of the function.
2. ***Evaluate the function*** at each such value.
3. ***Compute*** the left and right ***limits*** of the function at each such value.
4. If the function and the limits all exist and are equal, conclude that the function is ***continuous*** there.
5. If the equality $\lim_{x\to c} f(x) = f(c)$ does not hold, ***classify*** it by using the information so collected.

What do you mean by "classify?"

Figure out what kind of discontinuity it is, depending on the graphical feature it determines.

And how do we do that?

By working backwards: for now, we shall list the main types of discontinuity; in the next chapter we shall learn methods to compute limits that will allow us to classify discontinuities with certainty.

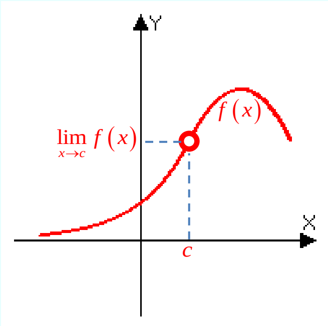
Since we have not seen those methods yet, for now I will only provide a simple example of each type and I will have to ask you to trust me even for them!

OK, let's see them.

Definition

A ***single point hole*** occurs when:

- $\lim_{x \to c} f(x)$ ***exists***, and
- $f(c)$ ***does not exist***.

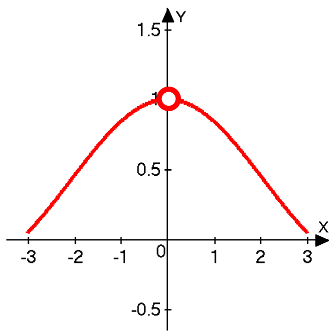


This means that the function is approaching a finite value, but does not reach it.

The resulting hole is usually represented by a small, hollow circle.

Example: $y = \dfrac{\sin x}{x}$

We have seen before that this function is not defined at $x = 0$, but the graphical and numerical methods suggest – and computational methods you have not seen yet will confirm – that the limit exists there and equals 1. Therefore we have a single point hole there.

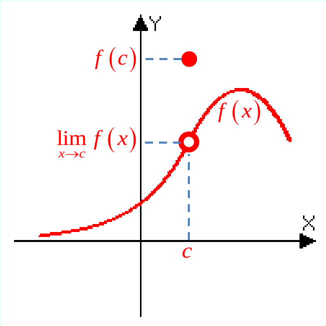


The next type of discontinuity is very artificial and it is used mostly to better illustrate the concept of discontinuity. Although there may be some practical examples of it, they are few and far between and you are not going to see some real ones until later, if ever. So, consider it as a kind of calculus calisthenics!

Definition

A ***displaced point*** occurs when:

- $\lim_{x \to c} f(x)$ and $f(c)$ ***exist***, but …
- they are ***not equal***.



This means that the function is approaching a finite value, but the value of the function is somewhere else.

The simplest way to create a displaced point is through a piecewise function.

Example:

$$\begin{cases} y = \dfrac{\sin x}{x} & if \;\; x \neq 0 \\ 0.5 & if \;\; x = 0 \end{cases}$$

This is a well defined function and, away from 0, it is the same as the previous one. But now it is defined at 0 as well, but not where we expect it. This generates a displaced point.

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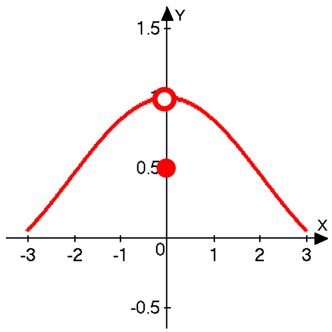


You can now see how contrived this situation is! There is an obvious connection between single point holes and displaced points that is highlighted by the following definition.

Definition

Single point holes and displaced points are also called ***removable discontinuities***, since the function can be made continuous at those values by changing a single function value and defining:

$$f(c)=\lim_{x\to c} f(x).$$

Example: $f(x)=\frac{\sin x}{x}$

We can make this function continuous by letting $f(0)=1$. We can do the same with the example of a displaced point, by changing the value of the function at 0 from 0.5 to 1:

$$f(x)=\begin{cases}\frac{\sin x}{x} & if\ \ x\neq 0\\ 1 & if\ \ x=0\end{cases}$$

The next two types of discontinuity should be very familiar, but I will mention them for completeness.

Definition

A ***vertical asymptote*** occurs when:

$$\lim_{x\to c^-} f(x)=\mp\infty \quad \text{or} \quad \lim_{x\to c^+} f(x)=\mp\infty.$$

Example: $y=\frac{\cos x}{x}$

This function is not defined at $x=0$ and the law of balloons tells us that the limit there is infinite, so that we have a vertical asymptote.

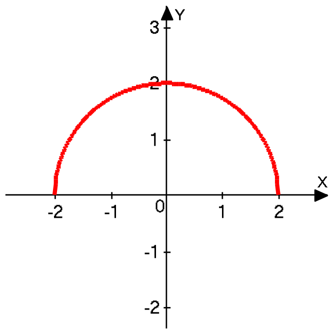
The next discontinuity is not considered such by some authors and teachers, as it may occur in functions that we would classify as continuous, according to our definition. However, I prefer to include it as a discontinuity, since it is consistent with the general idea of a discontinuity, meaning a break in the graph.

Definition

An ***end of domain*** discontinuity occurs when $f(x)$ is not defined on an interval to the left or to the right of the value.

Example: $y=\sqrt{4-x^2}$

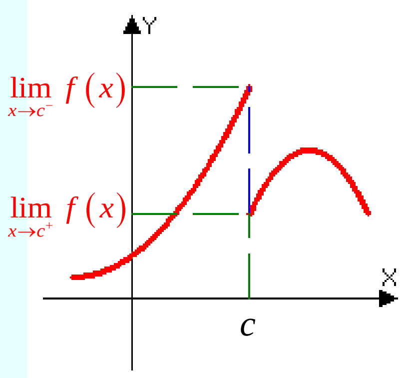
The domain of this function only extends from -2 to 2. So, both values provide end of domain discontinuities, since at each of them the function exists only on one side.



The last type of discontinuity that we shall consider may look artificial and rare, but it actually occurs often in real life, in the way that certain practical quantities are obtained.

Definition

A ***jump*** occurs when $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist, but are not equal.



In that case the graph is approaching two different, finite values, one from either side.

A jump usually occurs within a piecewise function.

Example: $\begin{cases} x^2 & if\ x<0 \\ 2-x & if\ x\geq 0 \end{cases}$

This function approaches 0 from the left, but 2 from the right, hence the jump.

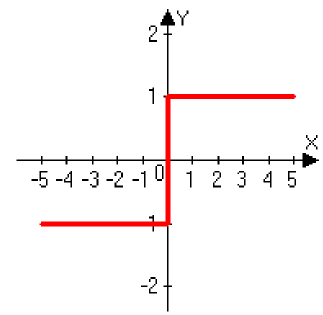
However, not every piecewise function has a jump!

Example: $\begin{cases} x^2+2 & if\ x<0 \\ 2-x & if\ x\geq 0 \end{cases}$

This function approaches 2 from both sides and it equals 2 at 0. Hence it is continuous there.

A jump can also occur in functions defined by a single formula. Remember this example from the previous section?

Example: $f(x)=\dfrac{\sqrt{x^2}}{x}=\dfrac{|x|}{x}$



This function is not defined at $x=0$. Moreover:

$\lim_{x\to 0^-} f(x)=-1$; $\lim_{x\to 0^+} f(x)=1$

This indicates a jump.

Remember how this function has fooled even a good computer program: the program insists on assuming the function to be continuous and tries to join the two pieces, creating a misleading graph.

Make sure not to be fooled by such situations!

Before closing this section, I need to point out that there are other types of discontinuity, but they are much rarer and almost pathological and, with very rare exceptions, will not show up in what comes next. In those few exceptions we'll discuss the situation separately.

Summary

- Discontinuities occur when continuity fails.
- The most useful and interesting aspect of a discontinuity is its classification into the specific type.
- The most common types of discontinuities encountered at our level are single point holes, vertical asymptotes and end of domain values.

Common errors to avoid

- Do not assume continuity or discontinuity at a value without checking the defining formula.
- Do not assume what type of discontinuity a function has at a value until you have checked its requirements.

Learning questions for Section D 1-7

Review questions:

1. Describe what *classifying a discontinuity* means.
2. List the major types of discontinuity and describe, for each, what identifies a discontinuity as being of that type.
3. Explain how to use limits to distinguish between a vertical asymptote and a removable discontinuity.

Memory questions:

1. Which graphical feature occurs when $\lim_{x \to a} f(x) = k$, but $f\left(a\right)$ does not exist?
2. Which graphical feature occurs when $\lim_{x \to c} f(x) \neq f(c)$, but both sides exist?
3. What is the name of a discontinuity that may occur when $\lim_{x \to c} f\left(x\right)$ exists?
4. Which two types of discontinuities are described as *removable*?

5. Which graphical feature occurs when $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, but both limits exist?

6. Which graphical feature occurs when $\lim_{x\to c} f(x) = \pm\infty$?

Computation questions:

For each of the functions presented in questions 1-28, identify the values at which a discontinuity may exist. Then use the graphical or numerical method to make a reasonable conjecture about the type of discontinuity. You will need more methods to compute limits in order to confirm these and other discontinuities.

1. $f(x) = \dfrac{x^2 + x + 12}{x - 3}$

2. $y = \dfrac{x+1}{x^2 - x}$

3. $f(x) = \dfrac{2x^3 - 128}{x^3 - 4x^2}$.

4. $y = \dfrac{x^2 - 5}{x^3 - 5\sqrt{5}}$

5. $f(x) = \dfrac{x - \sqrt{x}}{\sqrt{x} - 1}$

6. $y = \sqrt{2x^3 + 8x} - \sqrt{2x^3}$

7. $y = \dfrac{\sqrt[3]{8 - x} - 2}{x}$

8. $y = \dfrac{x - 3}{\sqrt{x^2 - 6x + 9}}$

9. $y = \dfrac{\sqrt{3x^2 + 5x}}{9x + 5}$

10. $y = \dfrac{3x}{\sqrt{1 - 4x^2}}$

11. $y = e^{\frac{1}{x^2 - 4}}$

12. $y = \dfrac{2}{e^{-x} + 5}$

13. $y = \ln\left(\dfrac{x^3 - x^2}{x - 1}\right)$

14. $y = \ln\left(\dfrac{x^3 + x^2}{x - 1}\right)$

15. $y = \ln\left(x^2 - x^3\right)$

16. $y = \ln\left(x^2 - 4\right)$

17. $y = \dfrac{\cos x - 1}{x \cos x}$

18. $y = \dfrac{\sec x - 1}{x \sec x}$

19. $y = \dfrac{\sin x}{\ln x^2}$

20. $y = \dfrac{3x}{\sin^{-1} 2x}$

21. $f(x) = \begin{cases} 2\sqrt{4 - x} & \text{if } x \le 4 \\ \dfrac{2}{4 + x} & \text{if } x > 4 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{3}{x - 5} & \text{if } x \le 2 \\ \dfrac{5}{x + 3} & \text{if } x > 2 \end{cases}$

23. $f(x)=\begin{cases} \dfrac{1}{2-x^2} & \text{if } x<0 \\ \dfrac{e^x}{2-x} & \text{if } 0\le x\le 1 \\ \dfrac{2}{1-x} & \text{if } x>1 \end{cases}$

24. $f(x)=\begin{cases} \dfrac{5-2x}{x-5} & \text{if } x<4 \\ 3 & 4\le x\le 5 \\ 2x-8 & x>5 \end{cases}$

25. $f(x)=\begin{cases} e^x & \text{if } x\le 0 \\ \sin x & \text{if } x>0 \end{cases}$

26. $f(x)=\begin{cases} e^x & \text{if } x\le 0 \\ \cos x & \text{if } x>0 \end{cases}$

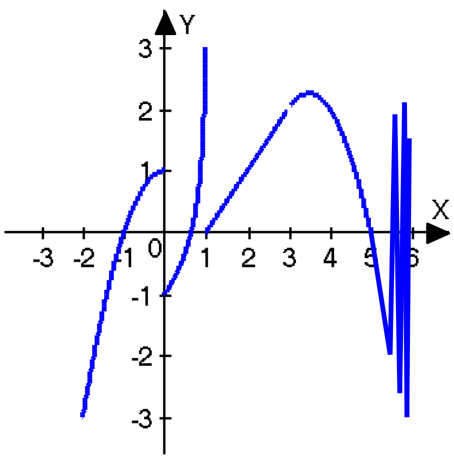
27. $f(x)=\begin{cases} \cos \pi x & \text{if } x<\dfrac{1}{3} \\ \sqrt{\dfrac{1}{2x}-1} & \text{if } x>\dfrac{1}{3} \end{cases}$

28. $f(x)=\begin{cases} \sec \pi x & \text{if } x<\dfrac{1}{3} \\ \sqrt{\dfrac{1}{x}+1} & \text{if } x>\dfrac{1}{3} \end{cases}$

29. $f(x)=\begin{cases} \dfrac{x+2}{x^2-4} & \text{if } x\le 0 \\ \dfrac{x-1}{2\cosh x} & \text{if } 0<x\le 1 \\ \dfrac{\sqrt{x-1}}{x^2-1} & \text{if } x>1 \end{cases}$

Theory questions:

1. Visually identify and tentatively classify all discontinuities of the function whose graph is shown here.



2. What are the three main graphical features that we have seen occurring at points of discontinuity?

3. What two conditions tell us that a function has a single point hole at $x=c$?

4. Which of the conditions for continuity is violated in a single point hole?

5. What two conditions tell us that a function has a displaced point at $x=c$?

6. Can a function be continuous at a value where a vertical asymptote exists?

7. If a function is defined at a value, can it be discontinuous there?

8. Do vertical asymptotes always correspond to discontinuities?

9. Is a jump considered a removable discontinuity?

10. When a given limit does not exist, can it correspond to a jump?

11. What kinds of function are more likely to contain a jump discontinuity?

12. Do all piecewise functions have discontinuities?

13. Can a function that is classified as continuous have a discontinuity?

14. Which equation is false at any discontinuity of a function?

15. The presence of what kind of asymptote indicates a discontinuity: horizontal, vertical, both or neither?

Proof questions:

1. If $f(x)$ is a function that is continuous at x=1, what graphical features can occur at x=1 in the graph of the function $y = \frac{f(x)}{x-1}$? Identify all possible options and explain how we can distinguish them by using limits.

Templated questions:

1. Identify and classify the discontinuities of any function on which you are working.

What questions do you have for your instructor?