Chapter 2: Resolving indeterminate forms — *Section 1*

Indeterminate forms

and some theoretical tools about them

What you need to know already:	*What you can learn here:*
➤ The concepts of limit and continuity.	➤ How to approach limit situations that are not clear at a first glance.

So far we have seen how to determine limits when they are *boring* – even though they may be irrelevant there – and when those simple "*laws*" involving infinity can be used. But that leaves out the most interesting situations, namely, those situations where something interesting occurs in the formula of the function, but we cannot see immediately what the limit turns out to be, or even if there is one!

We know that the graphical and numerical approaches provide only estimates, but no guaranteed conclusion. In this chapter we'll develop ***algebraic*** methods that, when used in the right situation, can provide the exact value of a limit.

And how do we know if the situation is right, so as not to waste our time?

Of course practice will play a big role, and you will need to experience the exhilarating feeling of being stuck and of realizing that the method you chose is not appropriate! But the point of this chapter is also to help you learn how to recognize the situation you are in. And here is the first important suggestion for that.

Knot on your finger

Even though the goal of a limit is to determine the behaviour of the function ***near*** a certain value, the ***first step*** in computing an unclear limit should always be to determine what form the function takes ***at*** that value.

Such a check can help us ***decide which method*** is more likely to work in the given situation.

Example: $\lim_{x\to 3}\frac{x^2-9}{x^3-27}$

This is not a boring limit, since the function is not defined at the value of interest, but if we try to evaluate it there, we get the form $\frac{0}{0}$, which is meaningless in itself, but, as we shall see soon, tells us which method to use to compute this limit.

So, by trying to evaluate the function, we get something that we cannot compute, but that tells us what to do next, right?

Absolutely. And that is why we call a form such as this *indeterminate*: it is not a computable quantity, but its structure gives us hints on how to proceed. And with this in mind, here is the definition of indeterminate form that we shall use.

Definition

A function $y = f(x)$ has an ***indeterminate form*** at $x = c$ (where *c* can be finite or infinite) if:

1. $f(x)$ is ***continuous*** on an interval including *c*, except possibly at *c*.
2. When we try to evaluate $f(x)$ at *c* we obtain ***one of the following forms***:

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\times\infty,\quad \infty-\infty,\quad 1^{\infty},\quad \infty^{0},\quad 0^{0}$$

Notice that here 0 and 1 represent ***variable*** quantities approaching that value, ***NOT*** constants with that value.

To ***resolve*** an indeterminate form, we need to use appropriate methods to determine if the limit exists and, if so, what its value is.

But if any number times 0 is 0, why is $0\times\infty$ indeterminate?

Read the definition again, but carefully! In these forms we are not dealing with the number 0, but with some variable quantity that is approaching 0. That makes all the difference in the world! To clarify what we are dealing with, let me explain why each of these forms is indeterminate.

Knot on your finger

The form $\frac{0}{0}$ is indeterminate because it represents the limit of the ratio of ***two very small quantities***. This ratio can approach any limit, depending on the relative size of the two quantities.

Example: $\lim_{x\to 0}\frac{x^2}{x^4}$; $\lim_{x\to 0}\frac{x^4}{x^2}$; $\lim_{x\to 0}\frac{ax^2}{x^2}$

Notice that in all three limits both numerator and denominator approach 0, so they all generate a $\frac{0}{0}$ form. However, the limits are different!

Since we are looking for the limit as $x\to 0$, we stay away from 0, so that we can divide top and bottom of each fraction by x^2.

Therefore the first limit is the same as $\lim_{x\to 0}\frac{1}{x^2}$, and the law of balloons tells us that this limit is ∞.

In the same way, the second limit is the same as $\lim_{x\to 0}x^2$, which equals 0.

Finally, $\lim_{x\to 0}\frac{ax^2}{x^2}$ becomes just *a*, whatever number *a* is.

Therefore, a 0/0 form is indeterminate because it can lead to any limit, and sometimes it is even possible that there is no limit!

Example:

$$\lim_{x\to 3}\frac{x-3}{x^2-9} \quad ; \quad \lim_{x\to 3}\frac{x-3}{\left(x^2-9\right)^2} \quad ; \quad \lim_{x\to 3}\frac{\sqrt{\left(x-3\right)^2}}{x^2-9}$$

As in the previous example, we can divide both numerator and denominator of each fraction by $\left(x-3\right)$ and in so doing we end up with three different limiting values: the first is $1/6$, the second is ∞, while the third does not exist, as it corresponds to a jump!

The method of dividing top and bottom by the same quantity is one of the methods we'll see in this chapter, and it is suggested by the 0/0 form.

Let me now deal with the indeterminacy of the other forms.

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The form $0\times\infty$ is indeterminate because it represents the limit of the product of a ***very large and a very small quantity***. This product can approach any limit, depending on the relative size of the two quantities.

Example:

We can use the same functions as in the last two examples, but thinking of them as product of the numerator and the reciprocal of the denominator:

$$\lim_{x\to 0}\frac{x^2}{x^4}=\lim_{x\to 0}\left(x^2\times\frac{1}{x^4}\right) \quad ; \quad \lim_{x\to 0}\frac{x^4}{x^2}=\lim_{x\to 0}\left(x^4\times\frac{1}{x^2}\right)$$

$$\lim_{x\to 3}\frac{x-3}{x^2-9}=\lim_{x\to 3}\left(x-3\right)\frac{1}{x^2-9}$$

and so on. If we try to evaluate them, we always get a $0\times\infty$ form and yet, as we have seen, these forms lead to different limits, hence they are indeterminate.

The next two cases follow the same rationale as the previous ones, so I will delay the corresponding examples until after we have seen some more methods to resolve indeterminate forms.

Knot on your finger

The form $\frac{\infty}{\infty}$ is indeterminate because it represents the limit of the ratio of ***two very large quantities***. This ratio can approach any limit, depending on the relative size of the two quantities.

Knot on your finger

The form $\infty-\infty$ is indeterminate because it represents the limit of the difference between ***two very large*** quantities. This difference can approach any limit, depending on the relative size of the two quantities.

The remaining forms involve powers and can be deceiving as they may remind you of certain familiar facts about power, except that those facts do not work in limit situations!

Unfortunately I will have to delay examples on these cases too, until we look at more methods to compute limits. For now just focus on the reasons given and bring to your instructor any doubts or puzzlements you may have.

Knot on your finger

The form 1^{∞} is indeterminate because it represents the limit obtained by raising a quantity ***close*** to 1 to a ***large*** exponent. Since large powers of a number greater than 1 become ever larger, while large powers of a quantity smaller than 1 become ever smaller, this form is indeterminate, as it can go either way to any value, including, of course, 1 itself.

You may think that raising 1 to any power always gives 1, which is true, but remember that here we are not raising 1 to a power, but a number *close* to 1. The relative size of base and power may lead to different limits

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The form ∞^{0} is indeterminate because it represents the limit obtained by raising a ***large quantity to a small exponent***. Since a small exponent makes the power of a large quantity smaller, base and exponent tend to opposite directions, so that this form can take any value, depending on the relative size of base and exponent.

This time you may think that any number to the 0 power is 1, but remember that we are not raising a number to 0, but a large *variable* quantity to a variable exponent *close* to 0. That makes all the difference and makes the form indeterminate.

Knot on your finger

The form 0^{0} is indeterminate because it represents the limit obtained by raising a ***small quantity to a small exponent***. Since a small exponent makes the power of a small quantity larger, base and exponent tend to opposite directions, so that this form can take any value, depending on the relative size of base and exponent.

For the last indeterminate form, notice also that $x^{0}=1$, but $0^{x}=0$ for any small positive number *x*, so would 0^{0} tend to 0 or 1? You can see that we are dealing with an indeterminate form.

I hope I have generated enough curiosity about these forms that you will be encouraged to learn the methods discussed in the next sections and thus see some really interesting limits for indeterminate forms.

As you can tell, this was mostly a conceptual section, so the *Learning questions* will reflect such theoretical emphasis.

Summary

- An indeterminate form occurs when the function would take on an expression that cannot be computed and for which the basic limit laws do not provide a definitive value.
- In that case we need other ways to compute the limit so as to determine what value the function is approaching.
- It is important to distinguish indeterminate forms, which must be resolved through appropriate and more advanced limit methods, from other non-numerical forms that can be evaluated by using basic limit laws.

Common errors to avoid

- Do not apply familiar properties of powers and rules of algebra to indeterminate forms!

Learning questions for Section D 2-1

Review questions:

1. Explain what an indeterminate form is.
2. Explain the difference between saying that a limit does not exist and saying that a limit produces an indeterminate form.
3. Explain why each of the indeterminate forms listed in this section is indeed indeterminate.

Memory questions:

1. What is the first step when analyzing an indeterminate form?
2. What are the four indeterminate forms involving arithmetic operations?
3. What are the three indeterminate forms involving powers?

Computation questions:

For each of the functions presented in questions 1-20:

a) Identify the values of x for which an indeterminate form occurs, if any.
b) Determine the type of indeterminate form it is.
c) Explain why it is indeterminate.

1. $y=\dfrac{t^3+8}{t+2}$

2. $y=\dfrac{s^3-1}{s^2-1}$

3. $y=\dfrac{5x^2}{3x^2-x}$

4. $y=\dfrac{x+2x^2}{x}$

5. $y=\dfrac{1}{\sqrt{4-x^2}}$

6. $y=\dfrac{x-3}{\sqrt{x^2-6x+9}}$

7. $y=\dfrac{\sqrt{4-z}-\sqrt{2}}{z^2-4}$

8. $y=\dfrac{1}{t-2}-\dfrac{4}{t^2-4}$

9. $y=(1+x)^{1/x}$

10. $y=\dfrac{x}{e^x-1}$

11. $y=e^{3/x}$

12. $y=e^{3x}$

13. $y=\ln\dfrac{2}{x+2}$

14. $y=\dfrac{\ln\left(x^2-1\right)}{x^2-4}$

15. $y=\dfrac{x\sin x}{1-\cos x}$

16. $y=\dfrac{\sin x}{\sqrt{x}}$

17. $y=\dfrac{\tan 7x}{\sin 3x}$

18. $y=\dfrac{\cos 2x+1}{x-\frac{\pi}{2}}$

19. $y=\begin{cases}\dfrac{3x-5}{x^2-1} & \text{if } x<0\\ \tan x & \text{if } 0\le x\le 2\\ \dfrac{x^2-2x-3}{x^2-3x} & \text{if } x>2\end{cases}$

20. $y=\begin{cases}\dfrac{2x^3+x^2+1}{x^2-1} & \text{if } x<1\\ \sqrt[3]{(x+7)^2} & \text{if } 1\le x\le 2\\ \dfrac{x^2-1}{x^2-9} & \text{if } x>2\end{cases}$

Theory questions:

1. Explain the form why 0^0 is not always equal to 1.

2. Explain why the form ∞^0 is not always equal to 1.

Provide a valid argument explaining why each of the forms presented in questions 3-7 is not indeterminate.

3. $0/\infty$

4. 0^∞

5. $\infty^{-\infty}$

6. $\frac{\infty}{0}$

7. ∞^1

8. In what sense can the form #/0 be considered indeterminate and why was it not listed as one of the indeterminate forms in this section?

Templated questions:

1. For any function you see, identify the values that may generate an indeterminate form, check if it is indeterminate and, if so, of what kind.

What questions do you have for your instructor?