Chapter 2: Resolving indeterminate forms — *Section 2*

The factor-and-cancel method

What you need to know already:	*What you can learn here:*
➤ What an indeterminate form is.	➤ The most common method for computing a limit of the form 0/0.

If a limit turns out to be of the indeterminate form 0/0, a very simple observation leads to a very effective method.

If a limit of the form $\lim_{x \to c} \frac{f(x)}{g(x)}$, where *c* can be finite or infinite, produces the indeterminate form 0/0, it is possible that the numerator and denominator have a common factor that becomes 0 at $x = c$, but not *near c*. Since when computing a limit we search for the behaviour of the function *near* the value *c*, not *at* that value, we can cancel that common factor and see what happens.

Strategy for computing a limit by factoring and cancelling a common factor

To try to compute the limit of an indeterminate form of the type 0/0:

- ➤ ***Factor*** numerator and denominator.
- ➤ ***Cancel*** any common factors that may be revealed in this way.
- ➤ ***Compute*** the limit of the function obtained after the cancelling.

If factoring is not possible, or if the new function produces another indeterminate form, a different method may be needed.

The wording of the strategy strongly suggests that this method works in many situations, but not always. Keep that in mind when you try to apply it.

Example: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

It is easy to check that this limit takes the form 0/0, so we can try to factor and cancel:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2}$$

We can see that the factor $x - 2$ is common and becomes 0 only at 2. Therefore we can cancel it, as it will not affect the computation of the limit, which looks at what happens *near* 2. This leads to:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)$$

The last limit is of the boring type and equals 4. Therefore, $\lim_{x\to 2}\frac{x^2-4}{x-2}=4$ and the function has a single point hole at $(2,4)$.

Example: $\lim_{x\to 3}\frac{x^3-3x^2-3x+9}{x^2-9}$

We begin by trying to evaluate the limit:

$$\lim_{x\to 3}\frac{x^3-3x^2-3x+9}{x^2-9}=\left(\frac{27-27-9+9}{9-9}\right)=\left(\frac{0}{0}\right)$$

This form suggests the presence of a common factor, so we factor both polynomials:

$$\lim_{x\to 3}\frac{x^3-3x^2-3x+9}{x^2-9}=\lim_{x\to 3}\frac{x^2(x-3)-3(x-3)}{(x-3)(x+3)}=\lim_{x\to 3}\frac{(x^2-3)(x-3)}{(x-3)(x+3)}$$

Now we cancel the common factor and evaluate the new limit, thus concluding that:

$$\lim_{x\to 3}\frac{(x^2-3)}{(x+3)}=\frac{9-3}{3+3}=1$$

Again, there is a single point hole, this time at $(3,1)$

Example: $\lim_{x\to 4}\frac{x^{-1}-0.25}{4-x}$

We begin by observing that this limit is worth computing, since at $x=4$ the denominator becomes 0. If we try to evaluate the function at that value we obtain:

$$\lim_{x\to 4}\frac{x^{-1}-0.25}{4-x}=\left(\frac{\frac{1}{4}-0.25}{4-4}\right)=\frac{0}{0}$$

Given this form, we try to factor and cancel. To do that, we first manipulate the numerator:

$$\lim_{x\to 4}\frac{x^{-1}-0.25}{4-x}=\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{4-x}=\lim_{x\to 4}\frac{\frac{4-x}{4x}}{4-x}=\lim_{x\to 4}\frac{4-x}{4x}\frac{1}{4-x}$$

We can now see the common factor and cancel it, thus concluding that:

$$\lim_{x\to 4}\frac{x^{-1}-0.25}{4-x}=\lim_{x\to 4}\frac{4-x}{4x}\frac{1}{4-x}=\lim_{x\to 4}\frac{1}{4x}=\frac{1}{16}$$

Once again, we have a single point hole, this time at $\left(4,\frac{1}{16}\right)$.

I read an interesting quote that seems to summarize what we are doing in this method: "*If you want to divide by zero, you have to sneak up on it from behind.*"

Cute! And I can see that when we apply the factor-and-cancel method we always end up with a single point hole.

CAREFUL! It may look this way, but don't make the common mistake of assuming this to be the case always!

Warning bells

A 0/0 form does ***not always*** identify a single point hole.

It does so ***only when*** the resulting ***limit exists and is finite***.

Example: $\lim_{x\to 2}\frac{2+x-x^2}{4-4x+x^2}$

As we try to evaluate this limit, we get the form 0/0, so we factor:

$$\lim_{x\to 2}\frac{2+x-x^2}{4-4x+x^2}=\lim_{x\to 2}\frac{(1+x)\cancel{(2-x)}}{(2-x)^{\cancel{2}}}=\lim_{x\to 2}\frac{(1+x)}{(2-x)}$$

But this is not a boring limit! It is of the form #/0, which indicates a vertical asymptote, not a single point hole.

Example: $\lim_{x\to 3}\frac{x-3}{\sqrt{x^2-6x+9}}$

By trying to evaluate the function at this value we obtain the form 0/0: is it a single point hole? We factor the denominator:

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x^2-6x+9}}=\lim_{x\to 3}\frac{x-3}{\sqrt{(x-3)^2}}=\lim_{x\to 3}\frac{x-3}{|x-3|}$$

By using the properties of the absolute value, we conclude that:

$$\lim_{x\to 3^-}\frac{x-3}{\sqrt{x^2-6x+9}}=-1,\qquad \lim_{x\to 3^+}\frac{x-3}{\sqrt{x^2-6x+9}}=1$$

And therefore this function has a jump, not a hole: always check a 0/0 form!

So, the method suggests a way to analyze the limit, but does not guarantee the conclusion!

Absolutely! Remember that the method does not always work and we cannot predict what it will reveal before we actually use it.

There is a little variant of this method that can be used in some situations. You may remember that factoring is very often a useful procedure, while its opposite, expanding, is useful less frequently. But it can assist in some situations.

Knot on your finger

In some situations where the factor and cancel method may seem be suitable, an ***initial step*** consisting of ***expanding*** some products may make the computations easier.

Example: $\lim_{x\to 0}\frac{(x-4)^2-16}{x}$

This limit produces a 0/0 form, whose top can be factored:

$$\lim_{x\to 0}\frac{(x-4)^2-16}{x}=\lim_{x\to 0}\frac{(x-4-4)(x-4+4)}{x}=$$

$$=\lim_{x\to 0}\frac{(x-8)\cancel{x}}{\cancel{x}}=\lim_{h\to 0}(x-8)=-8$$

But you may be intimidated by the unusual difference of squares! In that case, try expanding the square instead:

$$\lim_{x\to 0}\frac{(x-4)^2-16}{x}=\lim_{x\to 0}\frac{x^2-8x+16-16}{x}=$$

$$=\lim_{x\to 0}\frac{(x-8)\cancel{x}}{\cancel{x}}=\lim_{x\to 0}(x-8)=-8$$

Of course, if you take this approach, make sure that the expansion and all consequent steps are done correctly.

Summary

- Indeterminate forms of the type 0/0 may be resolved by factoring numerator and denominator and cancelling any common factors so obtained.
- This method does not always work, but it is very effective in may situations involving rational or other simple functions.

Common errors to avoid

- Do not assume that every indeterminate form of the type 0/0 corresponds to a single point hole.

Learning questions for Section D 2-2

Review questions:

1. Describe when and how to use the factor-and-cancel method.
2. Explain the connections and differences between the factor-and-cancel method and single point holes.

Memory questions:

1. In what situation is it appropriate to use the factor and cancel method to compute a limit?
2. Does an indeterminate form of the type 0/0 always correspond to a single point hole?

Computation questions:

Evaluate the limits presented in questions 1-12 and, in each case, determine which graphical feature is associated with it.

1. $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

2. $\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$

3. $\lim_{s\to 1} \frac{s^3-1}{s^2-1}$

4. $\lim_{x\to -1} \frac{x^3+3x^2+3x+1}{x^2+2x+1}$

5. $\lim_{x\to -4} \frac{x+4}{x^3+64}$

6. $\lim_{x\to 3} \frac{x^2-9}{2x^2-5x-3}$

7. $\lim_{x\to 2} \frac{2+x-x^2}{4-4x+x^2}$

8. $\lim_{x\to 2} \frac{x^2-4}{x^2+7x-18}$

9. $\lim_{h\to 0} \frac{(4+h)^{-2}-(4)^{-2}}{h}$

10. $\lim_{h\to 0} \frac{\frac{1}{4+h}-\frac{1}{4}}{h}$

11. $\lim_{x\to 4} \frac{2x^{3/2}-x^2}{2-\sqrt{x}}$

12. $\lim_{x\to 8} \frac{64-x^2}{2-\sqrt[3]{x}}$

For each of the functions presented in questions 13-25, identify where discontinuities may occur and, for each of them, use limits to classify them.

13. $f(x)=\frac{x^2+6x+5}{x^2+4x-5}$

14. $f(x)=\frac{x^3-4x}{3x^2-7x+2}$

15. $f(x)=\frac{2x^3-128}{x^3-4x^2}$.

16. $f(x)=\frac{x^3-3x^2+x-3}{x^2-4x+3}$

17. $f(x)=\frac{x^2-5}{x^3-5\sqrt{5}}$

18. $f(x)=\frac{x^3-2\sqrt{2}}{x^2-2}$

19. $f(x)=\frac{\sqrt[4]{x}-1}{x-1}$

20. $f(x)=\frac{\sqrt[3]{x}-1}{x-1}$

21. $f(x)=\frac{x^4-9}{x^2-\sqrt{12}x+3}$

22. $f(x)=\frac{|5x|}{2x}+\frac{6}{5}$

23. $f(x)=\begin{cases} \dfrac{x^3-1}{x^2-1} & \text{if } x<1 \\ 2 & \text{if } x=1 \\ x^2+1 & \text{if } x>1 \end{cases}$

24. $f(x)=\begin{cases} e^x & \text{if } x\le 2 \\ \dfrac{x^3-8}{x^2-4} & \text{if } x>2 \end{cases}$

25. $f(x)=\dfrac{e^{x+2}-xe^x}{e^4-x^2}$

26. $f(x)=\dfrac{x^3-4x^2}{2x^5-7x^4-4x^3}$

For the function provided in each of questions 27-28, determine the value, if any, that should be assigned to the constant *k* so that this function is continuous.

27. $f(x)=\begin{cases} \dfrac{x-2}{x^3-k} & \text{if } x<2 \\ \dfrac{2}{3k} & \text{if } x=2 \\ \dfrac{x-2}{3x^2-12} & \text{if } x>2 \end{cases}$

28. $f(x)=\begin{cases} \dfrac{x}{x^3-kx} & \text{if } x<0 \\ 2 & \text{if } x=0 \\ \dfrac{x-2}{x^2-2k} & \text{if } x>0 \end{cases}$

Theory questions:

1. When does a 0/0 indeterminate form correspond to a single point hole?

2. Provide a formula for a function that has a jump discontinuity at $x=2$ and a single point hole at $x=4$, but is continuous everywhere else. Explain why your choice works.

What questions do you have for your instructor?