Chapter 2: Resolving indeterminate forms — **Section 3**

The method of rationalizing

What you need to know already:	**What you can learn here:**
➤ The concept of limit and the factor-and-cancel method of resolving indeterminate forms.	➤ How to resolve indeterminate forms involving roots by rationalizing the function.

When the function whose limit we are trying to compute involves roots, factoring is usually not possible, but we can use special products to make the computations possible. First, let me remind you of what the process of rationalizing is in general.

Definition

To ***rationalize*** an expression means to multiply and divide it by a suitable factor so that the new version is easier to work with.

When the expression involves a ***sum or difference of square roots***, we can multiply and divide it by its conjugate and then use the ***difference of squares*** formula to change its appearance.

When the expression involves a ***sum or difference of cube roots***, we can multiply and divide it by the appropriate sum required by the ***sum or difference of cubes*** formula.

I think I need an example or two…

I agree, and here they are. However, you may also want to review the special product formulae related to differences and sums of square and cubes.

Example: $\dfrac{\sqrt{4-x}-2}{x}$

The numerator of this expression consists of a difference of square roots, since $\sqrt{4-x}$ is clearly such and $2=\sqrt{4}$. Therefore, we can rationalize it by multiplying and dividing by the conjugate of this difference, that is, the sum of the two roots:

$$\frac{\sqrt{4-x}-2}{x}=\frac{\sqrt{4-x}-2}{x}\times\frac{\sqrt{4-x}+2}{\sqrt{4-x}+2}$$

The top is now the product of a difference and its conjugate, so that the difference of squares formula applies and we can write:

$$\frac{\sqrt{4-x}-2}{x}=\frac{\sqrt{4-x}-2}{x}\times\frac{\sqrt{4-x}+2}{\sqrt{4-x}+2}=\frac{\left(\sqrt{4-x}\right)^2-2^2}{x\left(\sqrt{4-x}+2\right)}$$

We can now compute the squares, simplify the numerator and cancel as needed:

$$\frac{\sqrt{4-x}-2}{x}=\frac{4-x-4}{x\left(\sqrt{4-x}+2\right)}=\frac{-\cancel{x}}{\cancel{x}\left(\sqrt{4-x}+2\right)}=\frac{-1}{\sqrt{4-x}+2}$$

In this way we have eliminated the square root from the numerator. Of course by doing that we have created a square root in the denominator, but hopefully, this new version of the expression will help us in what we need to do, be it resolving indeterminate forms, or anything else

But aren't we supposed to rationalize always the denominator?

No. From your high school days, you may remember rationalizing as a method used primarily to eliminate roots from denominators. There is a historical and pedagogical explanation of this preference, but the method works for numerators as well. In fact, sometimes it may be good to rationalize even without starting from a fraction, as the next example shows.

Example: $\sqrt{4x^2+3x}+2x$

This expression consists of a sum of terms, one of which is a square root, so we rationalize it by using its conjugate, namely the difference of the two terms:

$$\sqrt{4x^2+3x}+2x=\left(\sqrt{4x^2+3x}+2x\right)\frac{\left(\sqrt{4x^2+3x}-2x\right)}{\left(\sqrt{4x^2+3x}-2x\right)}=\frac{\left(\sqrt{4x^2+3x}\right)^2-(2x)^2}{\sqrt{4x^2+3x}-2x}$$

Next we can expand and simplify the numerator:

$$\sqrt{4x^2+3x}+2x=\frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}-2x}=\frac{3x}{\sqrt{4x^2+3x}-2x}$$

Hopefully, this new version will be useful for what we need to do next.

Example: $\frac{\sqrt[3]{x+1}+1}{x^2-4}$

This time we have a cube root on top, so using the conjugate will not work, since squaring a cube root does not eliminate it. Instead we use the sum of cubes formula:

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$

Here we need $a=\sqrt[3]{x+1}$ and $b=1$, so that we can use:

$$\left(\sqrt[3]{x+1}\right)^3+1^3=\left(\sqrt[3]{x+1}+1\right)\left(\sqrt[3]{x+1}^2-\sqrt[3]{x+1}+1\right)$$

So, we multiply top and bottom by the last expression in brackets and see that:

$$\frac{\sqrt[3]{x+1}+1}{x^2-4}=\frac{\sqrt[3]{x+1}+1}{x^2-4}\frac{\left(\sqrt[3]{x+1}^2-\sqrt[3]{x+1}+1\right)}{\left(\sqrt[3]{x+1}^2-\sqrt[3]{x+1}+1\right)}=\frac{\left(\sqrt[3]{x+1}\right)^3+1}{\left(x^2-4\right)\left(\sqrt[3]{x+1}^2-\sqrt[3]{x+1}+1\right)}$$

Expanding and simplifying the top we conclude that:

$$\frac{\sqrt[3]{x+1}+1}{x^2-4}=\frac{x+2}{\left(x^2-4\right)\left(\sqrt[3]{x+1}^2-\sqrt[3]{x+1}+1\right)}=\frac{1}{(x-2)\left(\sqrt[3]{x+1}^2-\sqrt[3]{x+1}+1\right)}$$

And, once again, we hope that this will help us.

It seems to me that this is a long method that does not eliminate the root, so, how is it useful?

Knot on your finger

The method of rationalizing ***does not eliminate roots***. It only ***moves them*** to a different location in the expression and in a different form. It is useful only if the new form suits the required computations better.

Also, the new version of the expression may ***differ*** from the original one in its ***domain***, usually only for one value.

In fact, the difference in domain is what we can exploit when we apply the method of rationalizing to resolving indeterminate forms.

Strategy for resolving indeterminate forms involving roots

To resolve an indeterminate form involving ***roots***, apply the method of ***rationalizing*** so as to obtain a new function that does not have the original discontinuity, or whose form can be easily resolved.

Since in a limit we analyze the function near the value, not at it, the original function and the new version will have the same limit.

Example: $\lim_{x\to 0}\frac{\sqrt{4-x}-2}{x}$

This is a 0/0 form and we cannot factor the top, but we can rationalize it. By using what we have seen in a previous example, we obtain:

$$\lim_{x\to 0}\frac{\sqrt{4-x}-2}{x}=\lim_{x\to 0}\frac{-1}{\sqrt{4-x}+2}$$

But this is not indeterminate anymore, since the new version is defined at 0, in fact it is a boring limit! Therefore:

$$\lim_{x\to 0}\frac{\sqrt{4-x}-2}{x}=\lim_{x\to 0}\frac{-1}{\sqrt{4-x}+2}=-\frac{1}{4}$$

Rationalizing allowed us to change an indeterminate and puzzling limit to a really simple one. Notice that the original version did not include 0 in the domain, while the new one does, and that is what makes it work!

Example: $\lim_{x\to -\infty}\left(\sqrt{4x^2+3x}+2x\right)$

This is an indeterminate form of the type ∞-∞, but we cannot factor it to see what becomes of it. However, by rationalizing as we just did:

$$\lim_{x\to -\infty}\left(\sqrt{4x^2+3x}+2x\right)=\lim_{x\to -\infty}\frac{3x}{\sqrt{4x^2+3x}-2x}$$

This is now an ∞/∞ form and we can resolve it with the method of dividing top and bottom by the highest power of x. (Remember that we are really looking for a horizontal asymptote here.) Since x is negative we write $x=-\sqrt{x^2}$ and obtain:

$$\lim_{x\to -\infty}\frac{3x}{\sqrt{4x^2+3x}-2x}=\lim_{x\to -\infty}\frac{3}{-\sqrt{\frac{4x^2+3x}{x^2}}-2}=\lim_{x\to -\infty}\frac{3}{-\sqrt{4+\frac{3}{x}}-2}=-\frac{3}{4}$$

Hence this is the required limit and $y=-\frac{3}{4}$ is the left horizontal asymptote.

Example: $\lim_{x\to -2}\frac{\sqrt[3]{x+1}+1}{x^2-4}$

This is also a 0/0 form, and we rationalize it:

$$\lim_{x\to -2}\frac{\sqrt[3]{x+1}+1}{x^2-4}=\lim_{x\to -2}\frac{1}{(x-2)\left(\sqrt[3]{x+1}^2-\sqrt[3]{x+1}+1\right)}$$

Once again, this new function has no problems at -2, the last limit is boring and we can conclude that:

$$\lim_{x\to -2}\frac{\sqrt[3]{x+1}+1}{x^2-4}=\frac{1}{-4\times 3}=-\frac{1}{12}$$

Summary

- Indeterminate forms involving a ratio or a difference of roots may be resolved by rationalizing.

Common errors to avoid

- If you decide to square a function that involves a root, and do so properly, remember to square root the limit you obtain: you changed the function! In such case, it may be better to simply use anti-Murphy's law.
- Do not multiply by the conjugate when dealing with cube roots, since that will not eliminate the root, unless you add a mistake! Instead, use the sum or difference of cube formulae.

Learning questions for Section D 2-3

Review questions:

1. Describe when and how the method of rationalizing works.
2. Describe how to use the method of rationalizing when dealing with cube roots. To do that, you may want to review the factoring formulae involving sums and differences of cubes.

Memory questions:

1. What method should be considered to resolve an indeterminate form that involves roots?
2. Which formulae are needed when rationalizing an expression involving cube roots?

Computation questions:

Compute the limits presented in questions 1-22, by using the method of rationalizing when needed.

1. $\lim\limits_{x\to 2}\dfrac{\sqrt{4-x}-\sqrt{2}}{x^2-4}$

2. $\lim\limits_{x\to 2}\dfrac{(x+1)^2-9}{\sqrt{x-1}-1}$

3. $\lim\limits_{x\to -3}\dfrac{\sqrt{x+4}-1}{\sqrt{x+12}-3}$

4. $\lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3}{\sqrt{x-1}-1}$

5. $\lim\limits_{x\to 2}\dfrac{\sqrt{x^2-3}-1}{\sqrt{x^2+5}-3}$

6. $\lim\limits_{x\to 1}\dfrac{\sqrt{2x^2+7}-3}{\sqrt{3x^2-2}-1}$

7. $\lim\limits_{x\to 0}\dfrac{3-\sqrt{9-x}}{x}$

8. $\lim\limits_{x\to 4}\dfrac{\sqrt{8-x}-2}{x-4}$

9. $\lim\limits_{x\to 9}\dfrac{\sqrt{x-5}-4}{x^2-9x}$

10. $\lim\limits_{x\to 0}\dfrac{\sqrt{x-5}-4}{x^2-9x}$

11. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{8-x}-2}{x}$

12. $\lim\limits_{x\to 1}\dfrac{\sqrt[4]{x}-1}{x-1}$

13. $\lim\limits_{x\to 1}\dfrac{\sqrt{2x^2+7}-3}{3x^2-2x-1}$

14. $\lim\limits_{x\to 0}\dfrac{\sqrt{2x^2+9}-3}{3x^2-2x}$

15. $\lim\limits_{x\to 9}\dfrac{\sqrt{x-6}-\sqrt{3}}{\sqrt{x-5}-2}.$

16. $\lim\limits_{x\to 3}\dfrac{\sqrt{6-x}-\sqrt{3}}{\sqrt{12-x}-3}$

17. $\lim\limits_{x\to 5}\dfrac{\sqrt{10-x}-\sqrt{5}}{\sqrt{30-x}-5}$

18. $\lim\limits_{x\to -1}\dfrac{\sqrt{8-x}-3}{\sqrt{24-x}-5}$

19. $\lim\limits_{x\to\infty}\left(2x-\sqrt{4x^2+1}\right)$

20. $\lim\limits_{x\to\infty}\left(3x-\sqrt{9x^2-x}\right)$

21. $\lim\limits_{x\to\infty}\left(3x+\sqrt{9x^2-x}\right)$

22. $\lim\limits_{x\to-\infty}\left(3x+\sqrt{9x^2-x}\right)$

In each of questions 23-28, identify all discontinuities and asymptotes of the given function.

23. $y=\sqrt{2x^3+8x}-\sqrt{2x^3}$

24. $y=\sqrt{x^4+8x}-\sqrt{x^4}$

25. $y=\dfrac{x-3}{\sqrt{3x^2-7x}-\sqrt{6}}$

26. $y=\dfrac{x-4}{2\sqrt{7}-\sqrt{2x^2-x}}$

27. $y=\dfrac{3-x}{\sqrt{x^2+7}-4}$

28. $y=\dfrac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x}$

Theory questions:

1. Which basic algebraic method is behind the method of rationalizing?

Proof questions:

1. Use an algebraic method to estimate the value of $\lim_{x\to 0}\dfrac{\sqrt[3]{1+cx}-1}{x}$ in terms of *c*.

What questions do you have for your instructor?