Computing limit by

Combining fractions and logarithms

What you need to know already:	*What you can learn here:*
➤ Basic methods to resolve indeterminate forms.	➤ How to use properties of fractions and logarithms. ➤ How to resolve indeterminate forms by factoring and rationalizing.

The factor-and-cancel and rationalizing methods provide effective ways to resolve many common indeterminate forms. In other situations we cannot use them directly, but we can do so after combining terms in one of the following two ways.

Strategy for combining fractions of an indeterminate form

If $\lim_{x\to c} f(x)$ takes the form $\infty-\infty$ as a difference of two fractions both creating a #/0 form, the function can be modified by ***combining both fractions*** into a single one through a common denominator. After doing this, another method may be attempted to resolve the indeterminate form.

Strategy for combining logarithms of an indeterminate form

If $\lim_{x\to c} f(x)$ takes the form $\infty-\infty$ as a difference of two logarithms both approaching infinity, the function can be modified by ***combining both logarithms*** into one by using the formula

$$\ln a-\ln b=\ln\frac{a}{b} .$$

After doing this, another method may be applied to compute the resulting limit.

Example: $\lim\limits_{x\to -2}\left(\frac{x^2}{x+2}+\frac{2x}{x+2}\right)$

At first sight, we may be tempted to notice that:

$$\lim_{x\to -2}\frac{x^2}{x+2}+\frac{2x}{x+2}=\left(\frac{4}{0}+\frac{-4}{0}\right)$$

and that the two fractions will cancel each other, thus giving a limit of 0.

Beware! First of all, we are not sure of whether the two fractions go to opposite infinities or the same (we need to look at each side separately!) But ∞ is not even a number and therefore this kind of arithmetic is not valid.

Instead we can combine the original fractions before computing the limit:

$$\lim_{x\to -2}\frac{x^2}{x+2}+\frac{2x}{x+2}=\lim_{x\to -2}\frac{x^2+2x}{x+2}=\lim_{x\to -2}\frac{x(x+2)}{x+2}=\lim_{x\to -2}x=-2$$

Conclusion: since the function is undefined at -2, but has -2 as the limit there, it has a hole at (-2, -2).

Example: $\lim\limits_{x\to\infty}\left(\ln(x+3)-\ln(2x-5)\right)$

This is also an $\infty-\infty$ form, but we can combine the two logarithms:

$$\lim_{x\to\infty}\left(\ln(x+3)-\ln(2x-5)\right)=\lim_{x\to\infty}\ln\frac{x+3}{2x-5}$$

Since the logarithm is a continuous function, we can use anti-Murphy's law (nothing wrong happens here) and compute the limit inside it, so that:

$$\lim_{x\to\infty}\ln\frac{x+3}{2x-5}=\ln\left(\lim_{x\to\infty}\frac{x+3}{2x-5}\right)$$

Dividing top and bottom by the highest power, or using the law of the jungle, we conclude that:

$$\lim_{x\to\infty}\left(\ln(x+3)-\ln(2x-5)\right)=\ln\frac{1}{2}=-\ln 2$$

That's it, folks! It's just a simple trick, if you want, but one that will pop up frequently.

It is more profitable for you now to work on similar cases.

Summary

- Indeterminate forms resulting from adding several fractions or logarithms can be resolved by combining them into a single expression to which other methods can be applied.

Common errors to avoid

- Use proper algebraic steps when combining fractions and logarithms: no amount of limit prowess can fix a background algebra error.

Learning questions for Section D 2-4

Review questions:

1. Explain how to resolve an $\infty - \infty$ form that involves fractions or logarithms.

Memory questions:

1. What method can be tried to resolve an indeterminate form of the type $\infty - \infty$ that involves fractions or logarithms?

Computation questions:

1. Determine the graphical feature that occurs to $y = \frac{1}{x-2} - \frac{4}{x^2-4}$ at *x*=2.

2. Determine the graphical feature that occurs to $y = \frac{1}{x-2} - \frac{4x}{x^2-4}$ at *x*=2.

3. What are the horizontal asymptotes, if any, of the function $f\left(x\right) = \ln\left(x^2+3\right) + \ln\left(x-6\right) - 3\ln\left(3x-1\right)$?

4. What are the horizontal asymptotes, if any, of the function $f\left(x\right) = \ln\left(x^2+3\right) - 2\ln\left(3x-1\right)$?

5. What are the horizontal asymptotes, if any, of the function $f\left(x\right) = \ln\left(x+1\right) + 3\ln\left(x-2\right) - 4\ln\left(x-5\right)$?

6. Determine the vertical asymptotes, if any, of the function $f\left(x\right) = \log_2(x^2-4x) - \log_2(4-x) + 5$.

Theory questions:

1. Can we use the method of this section to compute $\lim_{x\to-\infty}\left(\ln\left(x+3\right) - \ln\left(2x-5\right)\right)$?

What questions do you have for your instructor?