Chapter 2: Resolving indeterminate forms — **Section 5**

Basic trigonometric limits

What you need to know already:	What you can learn here:
➤ The concept of limit, the squeeze theorem and the method of rationalizing.	➤ One fundamental limit involving limits and two basic methods for resolving simple limit situations.

We have already seen how to resolve certain limits involving trigonometric functions by using the informal laws of gravity, balloons and jungle. But there is one limit that cannot be resolved with any of them, and whose value we have only been able to estimate by using the numerical and graphical methods. Now shall now confirm that estimate a little more formally.

Technical fact:

The fundamental trigonometric limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

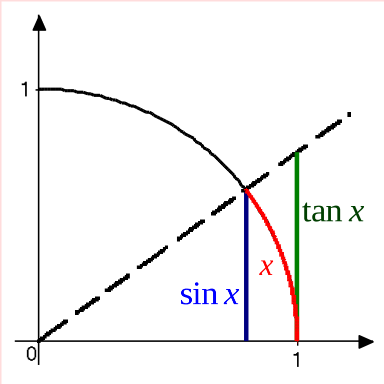
This limit is valid only when working in ***radians***.

Proof

This picture represents a small, positive angle of x radians and the corresponding arc (in red) whose length is, by definition of radians, also *x*.

The length of the left vertical segment (in blue) is then $\sin x$, while that of the right vertical segment (in green) is $\tan x$. As the picture shows:

$$\sin x < x < \tan x$$



Notice, however, that I am relying on a visual inspection, not a logical argument, to justify this inequality! There are strict logical ways to do so, but they are beyond our scope, so we'll just trust our eyes and the many mathematicians who have checked it!

Now, if we divide all parts of this inequality by $\sin x$, we obtain:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Since all three parts are positive, by taking their reciprocals we can see that:

$$\cos x < \frac{\sin x}{x} < 1$$

Since $\lim_{x \to 0} \cos x = 1$ (it is a boring limit!), the squeeze theorem assures us that $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$. A similar argument proves it from the other side.

Ok, we have now solved the mystery. So what?

We'll need this limit later for an important purpose: computing the derivative of the sine function. For now, we'll see what doors this limit opens. First, we can combine it with the method of rationalizing to get another important limit.

Technical fact

$$\lim_{x\to 0}\frac{\cos x-1}{x}=0$$

Proof

Although there are no roots here, we can still multiply and divide by the conjugate sum to resolve this 0/0 form:

$$\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\frac{\cos x-1}{x}\frac{\cos x+1}{\cos x+1}=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}$$

Now we use the basic Pythagorean identity $\sin^2 x+\cos^2 x=1$:

$$\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}=\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}$$

Finally we split the function into two factors and use the anti-Murphy's law:

$$\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}=-\lim_{x\to 0}\frac{\sin x}{x}\frac{\sin x}{\cos x+1}=$$

$$=-\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{\cos x+1}\right)=-1\times\frac{0}{2}=0$$

We can apply the same method of rationalizing and the basic Pythagorean identity to other similar limits containing a $\frac{\sin x}{x}$ factor and with x approaching 0.

And, of course, we can use the same approach when the variable is not x.

Knot on your finger

The basic limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ is valid for ***any quantity*** denoted by x, as long as the argument of the sine function and the denominator are equal. In particular, it is valid as:

$$\lim_{y\to 0}\frac{\sin y}{y}=1 \quad ; \quad \lim_{kx\to 0}\frac{\sin kx}{kx}=1 \quad ; \quad \lim_{f(x)\to 0}\frac{\sin f(x)}{f(x)}=1$$

Example: $\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}$

It is very tempting to cancel the "*sin*" and the "*x*" and conclude that the answer is $3/4$. But that would also be VERY bad mathematics, since "*sin*" is the name of a function, which cannot be cancelled and "*x*" is inside that function, and there is no valid way to take it out. So we need a different approach, namely the method of multiplying and dividing.

We start by dividing top and bottom by *x*:

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}=\lim_{x\to 0}\frac{\frac{\sin 3x}{x}}{\frac{\sin 4x}{x}}$$

Now, on the top and bottom we have *almost* the fundamental trig limit, except for the fact that the quantity inside the sine and the denominator are not equal. We can make them equal by using the multiply and divide procedure again:

$$\lim_{x\to 0}\frac{\frac{\sin 3x}{x}}{\frac{\sin 4x}{x}}=\lim_{x\to 0}\frac{3\frac{\sin 3x}{3x}}{4\frac{\sin 4x}{4x}}$$

Now we can split the limits, notice that as x approaches 0 so does $3x$ and complete the procedure:

$$\lim_{x\to 0}\frac{3\frac{\sin 3x}{3x}}{4\frac{\sin 4x}{4x}}=\frac{3\lim_{x\to 0}\frac{\sin 3x}{3x}}{4\lim_{x\to 0}\frac{\sin 4x}{4x}}=\frac{3}{4}\frac{\lim_{3x\to 0}\frac{\sin 3x}{3x}}{\lim_{4x\to 0}\frac{\sin 4x}{4x}}=\frac{3}{4}\frac{1}{1}=\frac{3}{4}$$

But this is the same value that you claimed not to be valid!

I did not say that the *value* was not valid, but that the naïve *method* to get it by cancelling *sin* and x was not correct! The value is the same, but getting a correct answer through an incorrect method is never acceptable!

Does that mean that I have to go through all these steps every time I need to compute such a limit?

If you are asked to demonstrate the method, yes! If, on the other hand, you only need to compute this limit within a longer, more complex solution, you can just remember the value of the limit and use it. When in doubt, ask for the expectations!

Here is another example where the fundamental limit can be used, even though at the beginning the situation looks different.

Example: $\lim_{x\to\pi}\frac{\sin x}{x-\pi}$

This limit leads to a 0/0 form, but x approaches π, not 0. Well, we can use the substitution $z=x-\pi$, so that our new variable z approaches 0 as x approaches π. In this way our limit becomes:

$$\lim_{x\to\pi}\frac{\sin x}{x-\pi}=\lim_{x\to\pi}\frac{\sin(x-\pi+\pi)}{x-\pi}=\lim_{z\to 0}\frac{\sin(z+\pi)}{z}$$

This is still NOT what we need, but by using a trigonometric identity, or the addition rule for sine, this limit becomes:

$$\lim_{z\to 0}\frac{\sin(z+\pi)}{z}=\lim_{z\to 0}\frac{-\sin z}{z}=-\lim_{z\to 0}\frac{\sin z}{z}=-1$$

Summary

- There is one fundamental trigonometric limit to memorize, namely $\lim_{x\to 0}\frac{\sin x}{x}=1$.
- Many other simple trig limits can be reduced to the fundamental one by rationalizing, substitution or other trig identities.

Common errors to avoid

- Do not cancel names of functions in an algebraic expression! Instead, always follow proper procedures involving identities and other methods.

Learning questions for Section D 2-5

Review questions:

1. State the fundamental trigonometric limit and explain why it is true.

2. Describe how to use the fundamental trigonometric limit to resolve other indeterminate limits that involve trigonometric functions.

Memory questions:

1. What is the value of $\lim_{x\to 0} \frac{\sin x}{x}$?

2. Which trigonometric limit should be considered as fundamental?

3. What is the value of $\lim_{x\to 0} \frac{\cos x - 1}{x}$?

Computation questions:

Compute the limits presented in questions 1-14 by using a suitable algebraic method.

1. $\lim_{x\to 0} \frac{\sin 2x}{5x}$

2. $\lim_{x\to 0} \frac{\sin 2x}{\sin 5x}$

3. $\lim_{x\to 0} \frac{\sin 2x}{\tan 5x}$

4. $\lim_{x\to 0} \frac{\sin 2x}{\cos 5x}$

5. $\lim_{\theta\to 0} \frac{\theta \cos\theta - \theta}{\sin^2 \theta}$

6. $\lim_{x\to 0} \frac{\cos 4x - 1}{3x}$

7. $\lim_{x\to \frac{\pi}{4}} \frac{\cos 2x}{4x^2 - \pi x}$

8. $\lim_{x\to \frac{\pi}{2}} \frac{\sin 2x}{2x^2 - \pi x}$

9. $\lim_{x\to \pi} \frac{\cos\left(\frac{x}{2}\right)}{\pi x - x^2}$

10. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\cos(2x)}{\pi x - 4x^2}$

11. $\lim\limits_{x\to 0} \frac{\sec x - 1}{x \sec x}$

12. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\tan x - 1}{x \tan x}$

13. $\lim\limits_{x\to 0} \frac{1-\cos 2x}{x^2}$

14. $\lim\limits_{x\to 0} \frac{1-\cos 3x}{2x^2}$

15. Determine the value of *a* for which the following function is continuous.

$$f(x) = \begin{cases} \frac{\sin ax}{x} & if\ x < 0 \\ \frac{27 - x^3}{ax + 9} & if\ x \geq 0 \end{cases}$$

16. Determine the value of *a* for which the following function is continuous.

$$f(x) = \begin{cases} \frac{\cos ax}{x-1} & if\ x < 0 \\ \frac{2 - x^2}{ax + 2} & if\ x \geq 0 \end{cases}$$

Identify and classify all discontinuities of the functions provided in questions 17-20.

17. $f(x) = \begin{cases} \frac{|x+1|}{x^2 - 1} & if\ x < 0 \\ \frac{\sin(-x)}{x} & if\ 0 < x \leq \pi \\ \frac{1-\cos x}{x} & if\ x > \pi \end{cases}$

18. $f(x) = x \sin \frac{1}{x}$

19. $f(x) = (x-2)\cot(3x-6)$.

20. $f(x) = \frac{\sin x}{2x^2 - \pi x}$

21. The site Spiked Math Comics asked the following question: what is $\lim\limits_{x\to k} \frac{sx^2 y \sin(k-x)}{k^2 - kx}$? Which common phrase provides the answer?

Theory questions:

1. What is the main purpose of knowing the fundamental trigonometric limit?
2. Which limit theorem is key to compute the fundamental trigonometric limit?
3. What is the value of $\lim\limits_{x\to\pi}\dfrac{\sin x}{x}$?
4. What is the value of $\lim\limits_{x\to 0}\dfrac{\cos x}{x}$?
5. It turns out that $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{\sin x^2}=1$, but we cannot prove that yet. So, what is wrong with this "*proof*" provided by a student?

$$\lim_{x\to 0}\frac{\sin^2 x}{\sin x^2}=\lim_{x\to 0}\frac{\sin x\cdot\sin x}{\sin x\cdot x}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

6. Let $u=f(x)$. Is it necessarily true that $\lim\limits_{x\to 0}\dfrac{\sin u}{u}=1$? Why or why not?

What questions do you have for your instructor?