Chapter 2: Resolving indeterminate forms — *Section 6*

Other transcendental limits

What you need to know already:	*What you can learn here:*
➤ Basic method for computing limits.	➤ A general method to deal with limits involving transcendental functions.

The methods for resolving indeterminate forms that we have seen so far go a long way, especially when it comes to situations appropriate at this level of your journey. But when the functions involved include more complicated transcendental portions, we may need to apply more knowledge and it may be useful to keep in mind a concept that is not new, but is worth employing on a regular basis.

Strategy for handling limits involving transcendental functions

When analyzing algebraically a limit that involves ***transcendental functions***, check for special features of those functions that can lead to a conclusion.

In particular, consider using any identities or special discontinuities, as well as information about limits at infinity.

Example: $\lim_{x\to 0} x^2 \sin\frac{1}{x}$

This limit is worth computing, since $x=0$ is the only value outside the domain of the function. We notice that as $x\to 0$, the first factor goes to 0 and the second is bounded between -1 and 1. Hence their product must become smaller and smaller. This means that the limit is 0.
Notice that arrive at our conclusion without doing any calculations, but only by using properties of the functions involved and basic logical arguments.

Example: $\lim_{x\to -\frac{\pi}{4}} \frac{\sin x+\cos x}{\cos 2x}$

When we try to evaluate the function at the given *x* value we obtain:

$$\frac{\sin\frac{-\pi}{4}+\cos\frac{-\pi}{4}}{\cos\frac{-\pi}{2}}=\frac{-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{0}=\frac{0}{0}$$

So we try to rationalize the numerator and use a double angle formula in the denominator to find a common factor:

$$\lim_{x\to -\frac{\pi}{4}} \frac{\sin x+\cos x}{\cos 2x}=\lim_{x\to -\frac{\pi}{4}} \frac{\sin x+\cos x}{\cos^2 x-\sin^2 x}\,\frac{\cos x-\sin x}{\cos x-\sin x}$$

$$= \lim_{x \to -\frac{\pi}{4}} \frac{\cos^2 x - \sin^2 x}{\left(\cos^2 x - \sin^2 x\right)\left(\cos x - \sin x\right)}$$

We now cancel and evaluate again for the conclusion:

$$= \lim_{x \to -\frac{\pi}{4}} \frac{1}{(\cos x - \sin x)} = \frac{1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

This time we used properties and identities of the trig functions involved, as well as some basic computations.

Example: $\lim_{x \to \infty} \frac{\tanh x - \tanh e^x}{\sinh x}$

This looks like a complicated limit, but it is very easy to evaluate once we remember some key properties of these hyperbolic functions, namely that:

$$\lim_{x \to \infty} \tanh x = 1 \quad ; \quad \lim_{x \to \infty} \sinh x = \infty$$

Now we can see that the numerator of the fraction approaches 0 and the denominator approaches ∞. Therefore:

$$\lim_{x \to \infty} \frac{\tanh x - \tanh e^x}{\sinh x} = \frac{0}{\infty} = 0$$

This time we used properties of the functions involved and our law of gravity, but again no calculations.

After we learn how to compute and use derivative, we shall meet another method to compute limits of transcendental functions. This is called ***L'Hospital's rule***, and you may have seen it before. In that case, remember that while it is a very efficient and effective method, it does not work every time, so that the methods of this chapter will continue to be useful and relevant.

Summary

- To compute limits involving transcendental functions, make use of known identities and properties of the functions involved.

Common errors to avoid

- Do NOT make up steps, formulae, limits and values. Instead, only use those that are true and verified. To help you do that, ask yourself if you are willing to bet $1,000 that your step is legitimate!

Learning questions for Section 2.6

Review questions:

1. Describe how to deal with limits involving transcendental functions.

Memory questions:

1. What is the value of $\lim_{x \to -\infty} \tanh x$?

2. What is the value of $\lim_{x \to \infty} \tanh x$?

Computation questions:

Compute the limits presented in questions 1-6 by using key properties of the functions involved.

1. $\lim_{x \to \infty} \sinh x$

2. $\lim_{x \to -\infty} \sinh x$

3. $\lim_{x \to \infty} \cosh x$

4. $\lim_{x \to -\infty} \cosh x$

5. $\lim_{x \to \infty} \tanh x$

6. $\lim_{x \to -\infty} \tanh x$

For each of questions 7-12, compute the given limit by using suitable identities and substitutions, as needed.

7. $\lim_{x \to -\frac{\pi}{4}} \frac{\sin x + \cos x}{\cos 2x}$

8. $\lim_{x \to 0} \frac{\cos x - 1}{\sin^2 x}$

9. $\lim_{x \to 0} \frac{\cos(\pi + x) + \cos x}{x}$

10. $\lim_{x \to 0} \frac{\cos(\pi + x) - \cos \pi}{x}$

11. $\lim_{x \to 1} \frac{\sin \pi x}{x - 1}$.

12. $\lim_{x \to \pi} \frac{\tan(\tan x)}{x - \pi}$.

For each of functions presented in questions 13-17, identify and classify all discontinuities and horizontal asymptotes, if any. Notice that you are expected to prove their existence by using limits, not just correctly guessing where they are. If the methods seen so far are not sufficient, conclude that more work is needed.

13. $f\left(x\right)=\dfrac{\sin x-1}{\cos^2 x}$

14. $f\left(x\right)=\dfrac{\cos x-1}{\sin^2 x}$

15. $y=\dfrac{\sin 3x}{e^{3x}-x-1}$ (NOTE: The denominator becomes 0 at $x\approx -0.94$ and another value that is easy to identify)

16. $f\left(x\right)=\dfrac{\sqrt{5-\cos^2 x}}{e^x-8}$

17. $f\left(x\right)=\begin{cases}\dfrac{x+2}{x^2-4} & if\ \ x\le 0\\ \dfrac{x-1}{2\cosh x} & if\ \ 0<x\le 1\\ \dfrac{\sqrt{x-1}}{x^2-1} & if\ \ x>1\end{cases}$

18. Determine the value of $\lim\limits_{x\to 0^+}\log_x\left(1+x^2\right)$ and explain why $\lim\limits_{x\to 0}\log_x\left(1+x^2\right)$ does not exist.

19. A function $f\left(x\right)$ is such that for any $x>10$, $\dfrac{\cosh x+5}{\sinh x+3}<f\left(x\right)<2-e^{\frac{7-x}{x^2}}$. What is its limit at infinity?

What questions do you have for your instructor?