Chapter 2: Resolving indeterminate forms **Section 7**

Limits for parametric and polar curves

What you need to know already:

- How to handle limits for functions of the standard form $y = f(x)$.
- How to use basic methods for resolving indeterminate forms.
- What parametric curves are and how they are defined.
- How to interpret the values of limits graphically.

What you can learn here:

- How to apply limit methods to the case of parametric curves.
- How to use limit procedures to identify certain graphical features of parametric curves.

Since a parametric curve is defined by using two regular functions, one for each coordinate, we can use all the methods we have developed so far to resolve indeterminate forms generated by each coordinate separately.

Example: $(x, y) = \left(\frac{\sin t}{2t}, \ln t^2 - \ln t \right)$

This curve is defined only for $t > 0$, since both coordinates become undefined at $t = 0$ and the logarithm is undefined for negative values. By using the methods we have seen so far, we see that for the x coordinate:

$$\lim_{t\to 0} x = \lim_{t\to 0} \frac{\sin t}{2t} = \frac{1}{2} \lim_{t\to 0} \frac{\sin t}{t} = \frac{1}{2}$$

For the y coordinate we focus on the right limit:

$$\lim_{t\to 0^+} y = \lim_{t\to 0^+} \left(\ln t^2 - \ln t\right) = \lim_{t\to 0^+} \ln \frac{t^2}{t} = \lim_{t\to 0^+} \ln t = -\infty$$

So, as t approaches 0, the x coordinate is finite and the y coordinate is infinite. As we now know, this indicates a vertical asymptote. You may want to check this on your graphing calculator.

So why do we need a whole separate section to discuss this?

Because of the last paragraph in the example!

I see: the usual computations tell us where x and y go separately, but we still need to classify the corresponding graphical feature, right?

Exactly: it is the curve that we are interested in! For a proper geometric interpretation of these limits we need to put the two limit values together and make sense of what they tell us.

Strategy for interpreting limits in a parametric curve

In order to classify the graphical features of a parametric curve by using limits, we need to ***look simultaneously*** at the limits generated by the two coordinates for the value of t in question.

The information provided by these limits, once combined, allows us to identify the nature of the feature under study. Here is a more practical way to look at this.

Technical fact

A parametric curve $\left(x(t), y(t)\right)$ has a ***single point hole*** for $t = c$ if:

- ***At least one*** of $x(c)$ and $y(c)$ ***does not exist***.
- ***Both*** $\lim_{t \to c} x(t) = a$ and $\lim_{t \to c} y(t) = b$ are ***finite***.

In such case the hole is at the point (a, b).

Example: $(x, y) = \left(\dfrac{t-5}{t^2-25}, \dfrac{\sin(5-t)}{t-5} \right)$

For this curve it is interesting to compute the limit as t approaches 5, since in that case both coordinates become undefined. By using the methods we have seen so far we can see that:

$$\lim_{t \to 5} \frac{t-5}{t^2-25} = \lim_{t \to 5} \frac{t-5}{(t-5)(t+5)} = \lim_{t \to 5} \frac{1}{t+5} = 0.1$$

$$\lim_{t \to 5} \frac{\sin(5-t)}{t-5} = -\lim_{t \to 5} \frac{\sin(5-t)}{5-t} = -\lim_{5-t \to 0} \frac{\sin(5-t)}{5-t} = -1$$

Therefore we can conclude that there is a single point hole at:

$$\lim_{t \to 5} \left(\frac{t-5}{t^2-25}, \frac{\sin(5-t)}{t-5} \right) = (0.1, -1)$$

Let us look at the less obvious issue you saw in the first example. How do we identify asymptotes for a parametric curve?

We look for where x or y go to infinity!

Yes, except that now x and y are both dependent on t, so we first need to figure out where to send t so that x or y will go to infinity.

Strategy for identifying asymptotes in a parametric curve

To identify a ***vertical asymptote*** of $\left(x(t), y(t)\right)$:

- ***Identify*** all values a for which $\lim_{t \to a} y(t) = \pm\infty$ even if from one side only
- For each such value, if $\lim_{t \to a} x(t) = c$ for a finite value c, then $x = c$ is a vertical asymptote.

To identify a ***horizontal asymptote*** of $\left(x(t), y(t)\right)$:

- ➤ ***Identify*** all values a for which $\lim_{t\to a} x(t) = \pm\infty$ even if from one side only.
- ➤ For each such value, if $\lim_{t\to a} y(t) = c$, as a finite value, then $y = c$ is a horizontal asymptote.

Too many infinities and letters! Example, please!

Example: $(x, y) = \left(\dfrac{1}{\sqrt{t^2 - 4}}, \ln t^2 \right)$

To identify the vertical asymptotes, we notice that the *y* coordinate, $\ln t^2$, becomes infinite when *t* approaches 0 or goes to $\pm\infty$.

For *t* near 0, *x* is not defined, so the issue is moot: there is no curve there!

For *t* approaching ∞ we have:

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{1}{\sqrt{t^2 - 4}} = 0$$

This means that, as *x* approaches 0, *y* is approaching infinity and therefore $x = 0$ is a vertical asymptote.

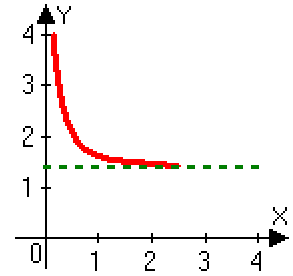
For $t \to -\infty$ the situation is the same, since the *t* in the formula is squared.

To find horizontal asymptote we need the *x* coordinate to go to ∞, which happens when *t* approaches -2 from the left or 2 from the right. In the latter case we have:

$$\lim_{t\to 2^+} y = \lim_{t\to 2^+} \ln t^2 = \ln 4$$

You can check that the same happens when *t* approaches -2 from the left, confirming that $y = \ln 4$ is a horizontal asymptote. Notice, however, that we are not looking at two asymptotes, but at one asymptote on the right side, since *x* goes to ∞ in both cases.

We can look at the the graph of the curve, which confirms our findings.



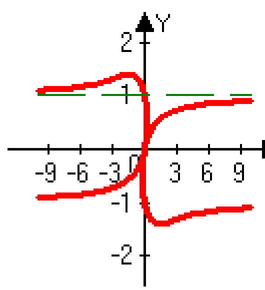
Notice that the computer program has difficulties in graphing the right side of the curve, as it requires large values of *t*. But that does not mean that the curve stops, only that the program can't handle it! Never trust completely a computer or calculator!

What about polar curves? Does the same approach work there as well?

Certainly! After all, each polar curve can be written in parametric form, so everything that can be done for parametric curves can also be done for polar ones, by considering their parametric form. And the main problem is still the same: how to identify asymptotes and/or holes.

Example: $r = 1 - \cot\theta$

If we graph this polar function we see a flattening of the graph towards $y = 1$ that looks suspiciously like a horizontal asymptote. In fact, some graphing calculators may include in this graph a line at $y = 1$:



Is there really a horizontal asymptote? Let us check.

First we notice that the formula of the curve suggests that the asymptote, if it exists, must occur when the radius becomes infinite, that is, when $\theta \to k\pi$, which is where the cotangent function becomes infinite.

We start by checking the *x* and *y* coordinates as $\theta \to \pi$. Their formulae are:

$$x = (1-\cot\theta)\cos\theta = \cos\theta - \cot\theta\cos\theta$$

$$y = (1-\cot\theta)\sin\theta = \sin\theta - \cos\theta$$

From the left we have:

$$\lim_{\theta\to\pi^-} x = \lim_{\theta\to\pi^-}(\cos\theta - \cot\theta\cos\theta) = -1-(-\infty)(-1) = -\infty$$

$$\lim_{\theta\to\pi^-} y = \lim_{\theta\to\pi^-}(\sin\theta - \cos\theta) = 0-(-1) = 1$$

This shows that as $\theta \to \pi^-$, *x* approaches -∞, while *y* approaches 1. But this is exactly what identifies a left horizontal asymptote at $y=1$.

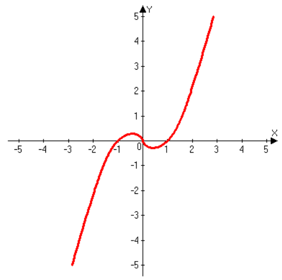
The same method can be used when approaching π from the right, thus confirming the finding, since as $\theta \to \pi^+$, *x* approaches ∞, while *y* approaches 1, thus generating the right horizontal asymptote that completes the proof.

The same method, when used as $\theta \to 0$, reveals the other visible horizontal asymptote at $y=-1$. You may want to go through those steps to check that.

Can we get single point holes in a polar curve?

Yes, they will occur for values of θ for which the curve is not defined, but the limits for both Cartesian coordinates exist as finite values.

Example: $r = e^{\tan\theta}$



This function is not defined for $\theta = \frac{\pi}{2}$, but since:

$$\lim_{\theta\to\frac{\pi}{2}^+} r = \lim_{\theta\to\frac{\pi}{2}^+} e^{\tan\theta} = \lim_{\tan\theta\to-\infty} e^{\tan\theta} = 0$$

the curve is approaching the only point with a polar radius of 0, that is, the pole itself. Therefore the pole is a single point hole for the function.

Notice that as $\theta \to \frac{\pi}{2}^-$, the radius goes to infinity, as the curve shows, but that does not affect the presence of the hole.

Summary

- To compute limits of parametric curves, simply compute the limit for each coordinate separately.
- To identify asymptotes, look for values of the parameter for which one coordinate becomes finite and the other infinite. Then interpret the results accordingly.

Common errors to avoid

- Do not confuse the variables involved and keep them in their proper roles as indicators of coordinates, be they Cartesian, parametric or polar.

Learning questions for Section D 2-7

Review questions:

1. Explain how to identify asymptotes and single point holes in parametric.

2. Explain how to identify asymptotes and single point holes in polar curves.

Memory questions:

1. How do we evaluate the limit of a parametric curve?

2. When does a curve $\left(x(t), y(t)\right)$ have a horizontal asymptote?

3. When does a curve $\left(x(t), y(t)\right)$ have a vertical asymptote?

Computation questions:

1. The parametric function $\left(\ln t, \dfrac{t}{t-2}\right)$ has three asymptotes. Identify them by using appropriate limits.

2. The parametric function $\left(\dfrac{\cos t}{t}, \dfrac{\sin t}{t}\right)$ has one asymptote. Identify it by using appropriate limits.

3. For what values of *a* and *b* is the following parametric function continuous?

$$(x, y) = \begin{cases} \left(e^t, (a+b)\cosh t\right) & if\ \ t \le 0 \\ \left(\dfrac{\sin at}{bt}, \dfrac{\cosh t}{t+1}\right) & t > 0 \end{cases}$$

4. Determine the value of $\left(x(2), y(2)\right)$ for which the parametric curve $\left(\dfrac{\sin(t-2)}{2t-4}, \dfrac{t^2-4}{t^2+t-6}\right)$ is continuous at $t = 2$.

Identify all asymptotes and single point holes of the curves presented in questions 5-10 by using suitable limits.

5. $\left(\frac{1}{e^{-t}-2}, e^{2t}\right)$

6. $\left(\frac{\cos t}{t}, \frac{\sin t}{t}\right)$

7. $\left(\frac{\sqrt{4t}-2}{t^2-3t+2}, \frac{t^2-1}{t-1}\right)$

8. $\left(\frac{\sqrt{4t}-2}{t^2-3t+2}, \frac{t^2-4}{t-4}\right)$

9. $r\theta = 1$

10. $r = 4 + 2\sec\theta$

11. $r = 3\frac{\cos 2\theta}{\cos\theta}$

12. $r = 1 - \cot\theta$

13. $r = \frac{3\theta - 1}{\theta}$

Theory questions:

1. Is it necessarily true that a polar curve $r(\theta)$ has an asymptote whenever *r* is approaching positive or negative infinity? Explain.

Templated questions:

1. Pick any parametric or polar curves and identify its single point holes and its asymptotes, if any.

What questions do you have for your instructor?