Chapter 3: Definition of derivative *Section 1*

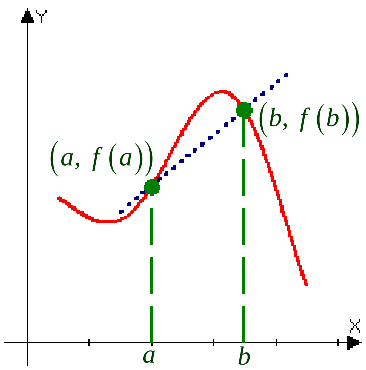
Derivative at a point

What you need to know already:	*What you can learn here:*
➤ The concept of limit and basic methods for computing limits.	➤ How to use limits to define the concept of slope of a function at a point, also known as the derivative.

Given any two points $(a, f(a))$ and $(b, f(b))$ on the graph of a function $y = f(x)$, we can always compute the slope of the line joining them by using the standard slope formula:

$$m = \frac{rise}{run} = \frac{f(b) - f(a)}{b - a}$$

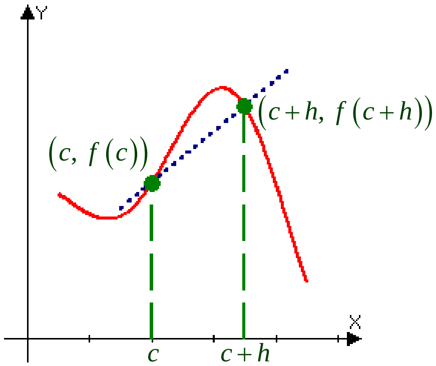
Such a line cuts through the graph at these two points, so that, borrowing from Latin, it is called the ***secant*** (cutting) line to the function at those points.



Also, this slope can be interpreted as the average rate of change of the function between $x = a$ and $x = b$, since it gives us the ratio between the total change in *y* and the total change in *x* over that interval.

So far, so good. Except that in many theoretical and, more importantly, applied problems the question of interest is not about an *average* slope or rate of change, but about the *instantaneous* slope or rate of change, that is, the slope at a *single* point. But in that case the slope formula is useless, since it gives the indeterminate form 0/0. Luckily, we now know how to handle such a form: by using limits!

To make the setting easier while working with limits, we reinterpret the secant line of the previous picture as the line joining a fixed point to another one nearby. To be consistent with the commonly used notation, I will now denote by *c* the *x* coordinate of the point of interest, by *h* an arbitrary small number and by *c*+*h* the *x* coordinate of a variable point near the one of interest, as shown in this graph.



With this notation, we can state and make sense of the following very important definition.

Definition

The ***slope of the secant line*** joining a point $\left(c, f\left(c\right)\right)$ on the graph of $y = f(x)$ to a nearby point $\left(c+h, f\left(c+h\right)\right)$ is given by:

$$m_{\text{sec}}\left(h\right) = \frac{rise}{run} = \frac{f\left(c+h\right) - f\left(c\right)}{h}$$

This slope can be interpreted as the ***average rate of change*** of $y = f(x)$ between $x = c$ and $x = c+h$.

Example: $y = x^2$ *at* $(1,1)$

If we apply the formula to this case, we obtain the slope:

$$m_{\text{sec}} = \frac{f\left(1+h\right) - f\left(1\right)}{h} = \frac{\left(1+h\right)^2 - 1}{h}$$

Example: $y = \dfrac{3}{x^2}$ *at* $\left(0.5, 12\right)$

We do the same with the given information:

$$m_{\text{sec}} = \frac{f\left(0.5+h\right) - f\left(0.5\right)}{h} = \frac{\dfrac{3}{\left(0.5+h\right)^2} - 12}{h}$$

Can't we simplify these expressions, or compute them for some values of h?

We certainly can, especially since we are assuming $h \neq 0$, but for now that is not needed and our real interest is in another issue that will come up soon. So, hold on to that thought for when it will become relevant.

Knot on your finger

In the definition of the ***slope of a secant line***, the quantity $m_{\text{sec}}\left(h\right)$ is a ***function of*** the small quantity $h \neq 0$, not of the *x* coordinate *c*, which we consider as a constant.

That real issue I just mentioned is the problem of determining the slope of the function *at* the point $\left(c, f\left(c\right)\right)$, without having to rely on an additional artificial point nearby. That is because the slope at the point, if we could compute it, tells us how fast the function is changing at that point, independently of our choice of the nearby point.

Technical fact

The slope of a function at a point can be interpreted as the ***instantaneous rate of change*** of the function at that point.

The first, natural attempt at computing the slope at a point would be to compute $m_{\text{sec}}\left(0\right)$. But this cannot be done, since in that case the denominator is 0. However, so is the numerator there and we are dealing with a 0/0 form!

So, we can compute the limit!

Well, we can try, since there is no guarantee that it will exist. But when it does, the limit is indeed the answer we are looking for. In fact, it is the starting point for the whole field of mathematics called calculus. And it is all based on that fundamental tool we call limits.

Definition

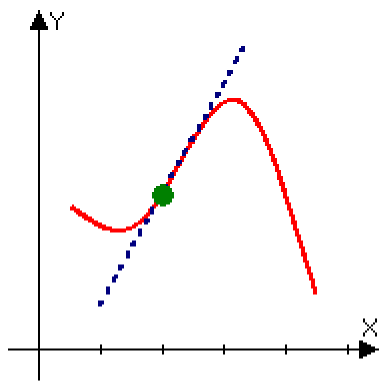
The ***slope*** of a function $y = f(x)$ ***at*** a point $(c,\ f(c))$ is given by the limit:

$$m_{\tan} = \lim_{h\to 0} m_{\sec}(h) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

The line with this slope through the point $(c,\ f(c))$ is called the ***tangent line*** to the function at $(c,\ f(c))$ and its equation is:

$$y = m_{\tan}(x-c)+f(c)$$

The picture we saw before for the secant line may now become something like this:



But this line does not look tangent in the way I am used to; in fact it seems to cut through the curve somewhere near our point!

Yes, and that is fine, since we expect the property of being tangent to hold at the point, not in a whole interval around it. We'll come back repeatedly to this issue and, hopefully it will become clearer. To be visually sure that this line is tangent we would need to look much closer to the point of tangency and our eyes may never be satisfied! That is why we base the definition on an algebraic property that can be checked exactly and without relying on visual acuity.

Example: $y = x^2$ *at* $(1,1)$

The slope of this point of the function is just the limit of the secant slope as $h\to 0$, that is:

$$m_{\tan} = \lim_{h\to 0} m_{\sec} = \lim_{h\to 0}\frac{(1+h)^2-1}{h}$$

By factoring and cancelling, we obtain:

$$m_{\tan} = \lim_{h\to 0}\frac{(1+h-1)(1+h+1)}{h} = \lim_{h\to 0}\frac{h(2+h)}{h} = \lim_{h\to 0}(2+h) = 2$$

The equation of the tangent line is therefore given by the point-slope formula:

$$y = m(x-x_0)+y_0 = 2(x-1)+1$$

Example: $y = \dfrac{3}{x^2}$ *at* $(0.5, 12)$

As before, we compute the limit of the secant slope, this time by combining fractions and then factoring and cancelling:

$$m_{\tan} = \lim_{h\to 0} m_{\sec} = \lim_{h\to 0}\frac{\dfrac{3}{(0.5+h)^2}-12}{h} = \lim_{h\to 0}\frac{\dfrac{12}{(1+2h)^2}-12}{h} =$$

$$= 12\lim_{h\to 0}\frac{1-(1+4h+h^2)}{(1+2h)^2}\frac{1}{h} = 12\lim_{h\to 0}\frac{-4h-h^2}{(1+2h)^2}\frac{1}{h} = 12\lim_{h\to 0}\frac{-4-h}{(1+2h)^2} = -48$$

We can conclude that the equation of the tangent line is:

$$y = -48\left(x-\frac{1}{2}\right)+12$$

Since the concepts of a tangent line and its slope are very important, several technical words and expressions have become associated with them and we now need to become acquainted with them.

Knot on your finger

The slope of a function at a point ***is the same*** as the slope of its tangent line at that point.

Therefore we shall use the expressions "*slope of a function*" and "*slope of the tangent line*" ***interchangeably***.

Definition

When the limit that defines the slope of a function $y = f(x)$ at a point $(c,\ f(c))$ exists, it is called the ***derivative*** of $y = f(x)$ at $x = c$.

We shall see more related jargon in the next section. For now, notice that the last three boxes can be summarized in one fact that is the most important thing that you need to remember, for life, from this calculus course:

Knot on your finger

Derivative = Slope = Rate of change

We'll see more of this in the next section. In particular we'll see that the derivative of the position function is the velocity and the derivative of the velocity function is the acceleration.

Before I show you another example, here is a summary of the steps we have taken so far.

Strategy for computing the derivative of a function at a point

In order to compute the derivative of a function $y = f(x)$ at a point $(c,\ f(c))$:

1. Compute $f(c+h)$ ***first***.
2. ***Insert*** such expression in the formula for the secant slope:

$$m_{\sec} = \frac{f(c+h) - f(c)}{h}$$

3. Use an appropriate method to ***evaluate the limit*** needed to obtain the tangent slope, the derivative:

$$m_{\tan} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

Example: $f(x) = \sqrt{x}$ *at* $(4, 2)$

To compute the derivative (slope) of this function at this point we observe that $f(4+h) = \sqrt{4+h}$, so that:

$$m_{\sec} = \frac{f(c+h)-f(c)}{h} = \frac{\sqrt{4+h}-2}{h}$$

The required slope is therefore:

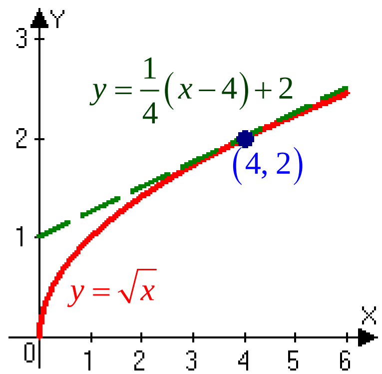
$$m_{\tan} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$$

This can be computed by rationalizing:

$$= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\,\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim_{h\to 0}\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$$

Therefore, the slope is $m = \frac{1}{4}$ and the tangent line is $y = \frac{1}{4}(x-4)+2$:



This looks like the method of "first principles"!

It is *exactly* what in high school is often called the method of *first principles*, since it is the method based on the defining formula

Why can't we use those quicker computational methods we also saw in high school then? They seemed much simpler and more practical.

You are referring to the so-called *"rules of differentiation."* We shall explore and use them soon and extensively. But first you need to be familiar with the definition derivative and how to use it, so as to fully understand this concept. This will greatly help you master later techniques and ideas that are based on it.

But that defining formula is complicated!

Only until you become used to it. In the meantime, to help you memorize it, keep in mind that it is the limit of a slope, that is, the limit of ***rise over run***!

And to complete this section, here is an alternative defining formula that some teachers and students prefer and that can be useful in some situations.

Technical fact

The following formulae provide ***equivalent*** and ***alternative*** definitions of the derivative of a function at a point $(c, f(c))$:

$$m_{\tan}(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$
$$= \lim_{x\to c}\frac{f(c)-f(x)}{c-x}$$

Proof

By performing the substitution $h = x-c$, the standard becomes the alternative:

$$m_{\tan}(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Multiplying numerator and denominator by -1 provides the second version.

Example: $f(x)=\sqrt{x}\ \ at\ \ (4,2)$

If we use the alternative definition, we get, also by rationalizing:

$$m_{\tan}(c)=\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\lim_{x\to 4}\frac{\sqrt{x}-\sqrt{4}}{x-4}=\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$=\lim_{x\to 4}\frac{x-4}{(x-4)\left(\sqrt{x}+2\right)}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

One advantage of this alternative definition is that it expresses the formula more clearly as the limit of a slope: the numerator is the rise and the denominator is the run. Also, it involves the original independent variable, x, without requiring the additional quantity h that, at least at the beginning, may be a bit of a mystery.

However, one advantage of the original definition is that it is computationally easier to work out in most of the situations we shall encounter. This is because in that formula the denominator consists of h only.

So, use whichever one you see fit and find more comfortable, but better still, become proficient in using both.

Summary

- The derivative of a function at a point is the slope of the tangent line there.
- To compute the derivative at a point, first compute the slope of a secant line joining the given point to a generic nearby point, then take the limit as the nearby point approaches the given one.
- The derivative can be interpreted also as the rate of change of the dependent variable with respect to the independent variable: ***DERIVATIVE=SLOPE=RATE OF CHANGE***

Common errors to avoid

- Use proper algebra to compute both the slope of the secant line and its limit as it becomes the tangent line. Poor use of algebra will lead you to dead ends or errors.

Learning questions for Section D 3-1

Review questions:

1. Explain why a limit is needed to compute the slope of a function.

2. Describe the meaning of all parts of the definition of the derivative of a function at a point.

3. Explain how the standard and alternative definitions of derivative at a point are related.

Memory questions:

1. What is the general formula that defines the derivative of a function $y = f(x)$ at $x = c$?

2. In the formula for the derivative of a function, what does the quantity *h* represent?

3. What is the geometrical meaning of the numerator of the formula for the derivative of a function at a point?

4. What is the geometrical meaning of the denominator of the formula for the derivative of a function at a point?

5. If a function represents a relationship between two practical quantities, what does its derivative represent in practice?

6. What is the *alternative* definition of the derivative of $f(x)$ that does not use *h*?

Computation questions:

For each of the functions and points given in questions 1-8, compute:

a) The formula for the slope of the secant line
b) The derivative at the point
c) The equation of the tangent line at the point

1. $y = x^2 + 2x - 5$ at (2, 3)

2. $f(x) = x^3 - x$ at $x = 2$.

3. $y = \dfrac{3}{x}$ at (2, 1.5)

4. $f(x) = \dfrac{1}{x-2}$ at $(3, 1)$

5. $y = \sqrt{x-5}$ at (6, 1)

6. $y=\sqrt{x^2+7}$ at $x=3$

7. $y=\left(\sqrt{x}+1\right)^2$ at (4, 9)

8. $f\left(x\right)=\dfrac{1}{\sqrt{x^2+9}}$ at $x=4$

9. $f\left(x\right)=\dfrac{x}{\sqrt{x^3+1}}$ at $x=2$.

10. $f\left(x\right)=\dfrac{x}{\sqrt{x^2+5}}$ at $x=2$.

The limit presented in each of questions 11-16 provides the derivative of a function $y=f\left(x\right)$ at $x=c$. Determine the function $f\left(x\right)$, the value c and the value of the derivative there.

11. $\lim\limits_{h\to 0}\dfrac{\left(h^2+3\right)^{1/2}-\sqrt{3}}{h}$.

12. $\lim\limits_{z\to \pi}\dfrac{\tan z}{z-\pi}$

13. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+kx}-1}{x}$

14. $\lim\limits_{x\to 1}\dfrac{e^x-e}{x-1}$

15. $\lim\limits_{x\to 1}\dfrac{\dfrac{x^3+1}{x}-2}{x-1}$

16. $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}$

17. The function $f\left(x\right)=\begin{cases} x^2\cos\dfrac{1}{x} & if\ \ x\neq 0 \\ 0 & if\ \ x=0 \end{cases}$ is continuous at *x*=0, as you may have checked in an earlier section. Use the definition of derivative to compute its derivative at *x*=0, thus showing that the derivative also exists there.

18. A function $f\left(x\right)$ has a derivative of 2 at $x=3$. Use this fact to compute $\lim\limits_{h\to 0}\dfrac{f\left(3+2h\right)-f\left(3-2h\right)}{h}$.

19. Estimate the derivative of the function $f(x)=x^3-x$ at $x=2$ by using the numerical method.

20. Estimate the derivative of the function $f\left(x\right)=\dfrac{1}{x-2}$ at $\left(3,1\right)$ by using the numerical method.

21. Estimate the slope of the line tangent to $f(x)=e^x$ at $x=0$ by finding the slope of *at least 3* suitable secant lines.

Theory questions:

1. What is the main connection between limits and derivatives?

2. The formula for the definition of derivative includes a fraction: what is its geometrical meaning?

3. Which procedure allows us to go from the slope of the secant line to that of the tangent line?

4. What can we conclude if we find out that $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ is not of the form 0/0?

5. How is the line tangent to the graph of $f(x)$ at those *x*-values for which its derivative is 0?

6. If a continuous function has a vertical tangent line at *x*=3, what can we say about its derivative there?

7. If the derivative of a function exists at a certain point, what kind of indeterminate form will be generated by the limit that defines it there?

8. Which derivative property is used to define the number *e*?

9. Identify one feature that the graph of a function may exhibit at a certain value of *x* where it not differentiable.

Proof questions:

1. Prove that if the derivative of $f(x)$ at $x=c$ exists, then $f(x)$ is continuous at $x=c$.

Templated questions:

1. Compute the derivative of any simple function at any point you deem suitable.

2. Compute any of the slopes required in the *Computation questions* by using the alternative definition.

What questions do you have for your instructor?