Chapter 3: Definition of derivative | Section 2

The derivative function

What you need to know already:	What you can learn here:
➤ What the derivative of a function $f(x)$ is at a point on its graph and how to compute it.	➤ How to define the derivative of a function, how to compute it by using such definition and how to denote it according to standard practice.

Despite what many people may think, one of the main goals of mathematics is to find the simplest possible way to solve a given problems. In particular:

Knot on your finger

A goal common to all areas of mathematics is the construction of a formula that can be used repeatedly to solve a certain type of problem, by using the known values of the key quantities involved.

Example:

Many centuries ago, mathematicians in several countries, working independently of each other, developed several ways to solve certain equations of the type that we now call quadratic. Most of these methods worked only in special cases, so it was desirable to find a general and simple way to solve *every* quadratic equation.

When the method of completing the square was developed in a clear way, it was apparent that it could be used to find a general method of solution for quadratic equations. This gave rise to what now is one of the most famous formulae in mathematics, the quadratic formula. It's a single, simple formula that allows us to solve *every* quadratic equation.

If you are interested in seeing how completing the square leads to the quadratic equation, check the note on Completing the square.

Cool, but why are you saying this now?

In the previous section we looked at how to compute the slope of a function at a point. But why should we go through all that work just to find the slope at *one* point? If we could find a single way to find the derivative of a function at *every* point, it would be more efficient, no? But to do that we need to develop a method that works exactly in the kind of general way I just mentioned.

It turns out that in our case such general formula is very simple to obtain. All we have to do is use the same method we used to get the slope at a point, but leave the x coordinate variable.

Definition

Given a function $y = f(x)$, its ***derivative function*** or simply its ***derivative***, is defined as the function:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This new function is identified verbally as ***f-prime***.

Although the definition is exactly the same as that for the derivative at a point, only with *c* changed to *x*, an important difference separates the two definitions.

Knot on your finger

- The derivative of a function at a point provides a ***single number*** that represents the slope, or rate of change, of the function there.
- The derivative function is a whole ***new function*** whose output values represent the slope, or rate of change, of the original function at *every* point where such function has a slope.

So, do we compute the derivative function with the same strategy as for the derivative at a point?

Yes, except for the use of the variable instead of the constant.

Example: $y = x^2$

In the previous section we saw how to compute the slope of this function at $(1,1)$. But why waste all that energy on one point only when with the same effort we can get ALL slopes at all points where there is a slope?

So, let's do it more generally. We start by computing $f(x+h)$

$$f(x+h) = (x+h)^2$$

We then include it in the defining formula and compute the limit:

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{\not{h}(2x+h)}{\not{h}} = \lim_{h\to 0}(2x+h) = 2x$$

So, if we need to know the slope of this function at any point (c, c^2), we just need to compute $2c$. Very efficient!

But don't we get to face more complicated calculations because of using x instead of a number?

Yes, but, as you are probably aware¸ we shall soon develop much faster ways to compute this new function, methods that do not require the computation of complicated limits. However, you still need to understand the general definition and apply it to relatively simple functions, such as this one.

Example: $g(x) = \dfrac{3}{x^2}$

To determine the derivative of this function , we apply the definition:

$$g'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{3}{(x+h)^2}-\dfrac{3}{x^2}}{h}$$

We change the numerator of the fraction by combining its own fractions:

$$=3\lim_{h\to 0}\frac{x^2-(x+h)^2}{x^2(x+h)^2}\frac{1}{h}=3\lim_{h\to 0}\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}\frac{1}{h}$$

$$=-3\lim_{h\to 0}\frac{h(2x+h)}{x^2(x+h)^2}\frac{1}{h}=-3\lim_{h\to 0}\frac{2x+h}{x^2(x+h)^2}=-\frac{6}{x^3}$$

Therefore the derivative of $g(x)=\frac{3}{x^2}$ is $g'(x)=-\frac{6}{x^3}$.

Example: $f'(x)=\sqrt{x^2+1}$

To compute the derivative of this function, we use the definition:

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x^2+2xh+h^2+1}-\sqrt{x^2+1}}{h}$$

We then rationalize:

$$=\lim_{h\to 0}\frac{\sqrt{x^2+2xh+h^2+1}-\sqrt{x^2+1}}{h}\frac{\sqrt{x^2+2xh+h^2+1}+\sqrt{x^2+1}}{\sqrt{x^2+2xh+h^2+1}+\sqrt{x^2+1}}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{x^2+2xh+h^2+1}+\sqrt{x^2+1}\right)}$$

$$=\lim_{h\to 0}\frac{2xh+h^2}{h\left(\sqrt{x^2+2xh+h^2+1}+\sqrt{x^2+1}\right)}=\frac{2x}{2\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}$$

Therefore the slope of this function at $x=1$ is $f'(1)=\frac{1}{\sqrt{2}}$, while at $x=-2$ it is $f'(-2)=-2/\sqrt{5}$.

So, this is also the method called "first principles."

Yes, although I still prefer to describe it as *the method based on the definition*, both because it is not clear what first principles we are referring to, and to remind you that this method relies on what the derivative *is*, rather than on short cuts that may obscure the meaning of what we are doing.

Because of the importance of derivatives, a whole set of words have been developed to describe particular aspects of them. We'll see most of them in later sections, but some of them need to be seen, used and spelled correctly right away.

Definition

If the derivative of $f(x)$ exists at $(c, f(c))$, we say that $f(x)$ is ***differentiable*** at that point.

If $f'(x)$ exists at every point in the domain of $f(x)$ we simply say that $f(x)$ is ***differentiable***.

The process of computing derivatives is referred to as ***differentiation***. To engage in such process is described by the verb ***to differentiate***.

The property for a function to be differentiable is called ***differentiability***.

By using these words, we can state in a short way a very important property of derivatives that will be used very often in later sections.

Technical fact

If a function is differentiable at a point, then it is continuous there:

Differentiability implies continuity

It may be stated simply, but I am not sure of what it means!

To make sense of this short sentence, remember that if the derivative at a point exists (differentiability), then the function has a slope there (slope and derivative are the same thing) and hence a tangent line. But in that case the function cannot have a discontinuity (a break) otherwise how could we construct such tangent line? An asymptote or a removable discontinuity would not allow us to attach the tangent line, while a jump would require a sudden step, not a gradual increase.

We can also clarify this concept by using the defining formula for the derivative, as follows.

In the indeterminate form $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ the denominator approaches 0, so the numerator must also approach 0, or we would get a #/0 form that leads to infinity, not to a finite value. This means that:

$$\lim_{h\to 0}\left[f(c+h)-f(c)\right]=0 \quad\Rightarrow\quad \lim_{h\to 0} f(c+h)=f(c)$$

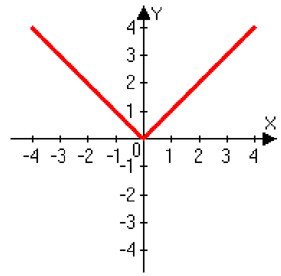
If we let $x=c+h$ this becomes $\lim_{x\to c} f(x)=f(c)$, which is the condition for continuity. Therefore, the presence of differentiability implies the presence of continuity.

Please notice that this may be a convincing algebraic argument, but it hides some subtle technical details that mathematicians have checked, so you should not take it as a proof, but you can trust its substance.

On the other hand, isn't this fact rather redundant? Isn't it also true that every continuous function is differentiable?

Absolutely NOT! This is an easy mistake to make, to the point that some early mathematicians assumed it, but it is not true.

Example: $f(x)=|x|=\sqrt{x^2}$



The graph of the absolute value function is shown here and you can see that it consists of two half-lines joining at the origin. The graph has no breaks and you can check formally that it is continuous everywhere, including at the origin. But you can clearly see (and maybe check formally) that it has no clearly defined slope at the origin; hence it is not defined there.

Technical fact

A function may be continuous at a point without being differentiable there.

Continuity does NOT imply differentiability

In the learning activities you will have a chance to discover other simple functions that are continuous, but not differentiable at all points.

On a different topic, why is the "prime" notation used for derivatives?

Besides its visible connection to the function from which it comes, it is short and it can be used and generalized effectively when we start playing with more advanced concepts, such as higher derivatives.

However, there are other notations for the derivative, that can be used, are used and, in certain situations, are more effective that the standard prime notation. Here are the most common and used ones.

Definition

The derivative of a differentiable function $y = f(x)$ may be denoted by using:

- The ***functional*** notation: $y = f'(x)$
- The ***short*** notation: y'
- The ***fractional*** notation: $\dfrac{dy}{dx}$ or $\dfrac{df}{dx}$
- The ***operator*** notation: $\dfrac{d}{dx} f(x)$
- The ***fluxion*** notation: $\dot{y}$

That is cool, but confusing! Who came up with them and why?

Some famous and important mathematicians came up with them and usually for some very good reason that we shall not discuss here, although you will see some of those reasons soon.

And just to drop some names:

- The functional and short notations are due to J.L. Lagrange.
- The fractional notation is dues to G. Leibniz and is in fact known as the ***Leibniz*** notation.
- The operator notation is a variant of one that goes back to J. Bernoulli.
- And the fluxion notation was invented by the man himself, Isaac Newton, but is now rarely used, together with the "*fluxions*" terminology. You may find it in physics when the independent variable is time.

Example: $f'(x) = \sqrt{x^2+1}$

We have seen that for this function $f'(x) = \dfrac{x}{\sqrt{x^2+1}}$. This is in the functional notation, but we can also write:

- Short notation: $y' = \dfrac{x}{\sqrt{x^2+1}}$
- Leibniz notation: $\dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{x}{\sqrt{x^2+1}}$
- Operator notation: $\dfrac{d}{dx} f(x) = \dfrac{x}{\sqrt{x^2+1}}$
- Newton's notation: $\dot{y} = \dfrac{x}{\sqrt{x^2+1}}$

We shall stop here for now: work on understanding the concept and the associated terminology and notation. You can also try a few more computations by using the definition, but we shall soon develop and use much better computational methods.

Summary

- The derivative of a function $f(x)$ is a new function, denoted by $f'(x)$, whose output values represent the slope, or rate of change, of the original function at the corresponding values of *x*.
- The defining formula, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is a very important formula that must be understood and memorized.

Common errors to avoid

- A key step in the computation of the derivative function is to *correctly* obtain an expression for $f(x+h)$. If this is not done right, all other subsequent steps, and the conclusion, may be incorrect.

Learning questions for Section D 3-2

Review questions:

1. Explain the relation between a function and its derivative.
2. Explain what "*differentiability implies continuity*" means.
3. Present a reasonable argument of why differentiability implies continuity, but not the reverse.

Memory questions:

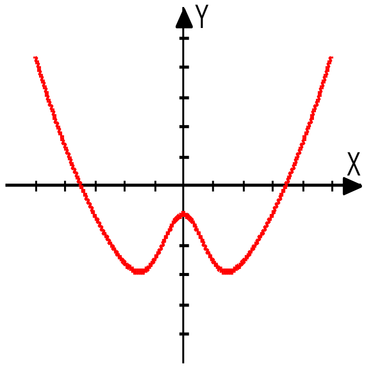
1. In the standard formula for the derivative of a function, what does the quantity *h* represent?
2. Which verb describes the activity of computing derivatives?
3. What is the correct Leibniz notation for the derivative of a function $y=f(x)$?
4. What is the short notation for the derivative of a function $y=f(x)$?
5. What is the operator notation for the derivative of a function $y=f(x)$?
6. If a function is differentiable, is it also continuous?
7. If a function is continuous, is it also differentiable?
8. Which technical noun identifies the process of computing derivatives? Make sure to spell it correctly!

Computation questions:

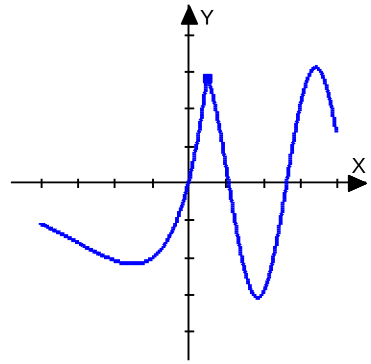
Use the defining formula to compute the derivative of the functions presented in questions 1-24.

1. $f(x)=x^2-x$
2. $f(x)=12x+7x^2$
3. $f(x)=6x+17x^2$
4. $y=\dfrac{1}{x+2}$
5. $y=12+\dfrac{7}{x}$
6. $y=\dfrac{3}{x^2+1}$
7. $f(x)=\dfrac{3x}{x^2+1}$.
8. $f(x)=x^2+\dfrac{2}{3+x}$
9. $f(x)=\left(1-\dfrac{x}{8}\right)^2$
10. $f(x)=\left(1-\dfrac{x}{2}\right)^3$
11. $f(x)=\sqrt{x-5}$
12. $y=\sqrt{2x-5}$
13. $f(x)=\sqrt{2-3x}$
14. $y=\left(1-x^2\right)^{\frac{1}{2}}$.
15. $y=\sqrt[3]{x}$
16. $y=\sqrt{x^2+3}$
17. $f(x)=\dfrac{2}{\sqrt{3-x}}$
18. $f(x)=\dfrac{6}{x}+7\sqrt{x}$
19. $f(x)=\dfrac{2}{\sqrt{x^2+x}}$
20. $f(x)=\dfrac{1}{\sqrt{x-2x^2}}$
21. $f(x)=\dfrac{1}{\sqrt{x^2+2x}}$
22. $f(x)=\dfrac{1}{\sqrt{x^2+3}}$
23. $y=\left(\sqrt{x+1}+2\right)^2$
24. $y=\sqrt{x+1}+\left(x+1\right)^2$

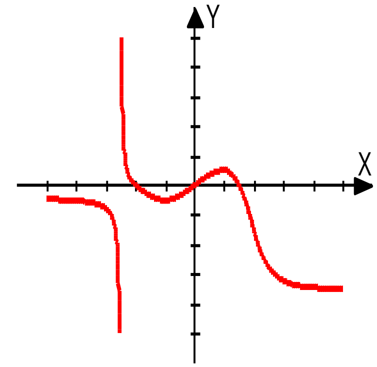
In questions 25-27, the graph of a function $y = f(x)$ is given. Sketch a graph of $y = f'(x)$ on the same coordinate frame, and explain how you obtained such graph.



25.



26.



27.

Theory questions:

1. Is it possible for a function to be continuous and for its derivative to have a jump?

2. Is it possible for a continuous function to have a vertical tangent line?

3. What do the *x*-intercepts of the derivative represent for the original function?

4. Does the derivative function represent the tangent line to a function?

5. Can two different functions have the same derivative?

6. If a function has a jump at *x*=*c*, what can we say about its derivative there?

7. What features of the ***graph*** of $f(x) = \sqrt{x-5}$ justify the fact that the graph of $f'(x)$ is entirely in the first quadrant?

8. What can we say about the derivative of a function at a point where its graph is going down?

9. Are there more functions that are continuous at *x*=3 or more functions that are differentiable there?

10. If a function has a removable discontinuity at *x*=*c*, is it differentiable at *x*=*c*?

Proof questions:

1. Use the definition of derivative to show that the derivative of $f(x)=x^{3/5}$ does not exist at $x=0$.

Templated questions:

1. Use the defining limit, but no differentiation rules, to compute the derivative of any simple function.
2. Determine the domain of any derivative you compute and compare it to the domain of the original function.

What questions do you have for your instructor?