Chapter 3: Definition of derivative *Section 3*

Interpretation and applications of derivatives

What you need to know already:	*What you can learn here:*
➤ The definition of derivative at a point and as a function.	➤ How derivatives are used in the most common applications.

We have already seen that, because of the way they are defined, derivatives can be naturally interpreted in two key ways. In this section we shall focus on both interpretations, starting from the first one, always a good place to start!

Knot on your finger

The ***graphical*** interpretation of the derivative of a function $f(x)$ at a value $x=c$ is that it represents the ***slope of the curve***, meant as the ***slope of line tangent*** to the graph of $f(x)$ at $(c, f(c))$.

In fact, the tangent line exists if and only if $f'(c)$ exists.

This observation gives us an obvious idea for using derivatives in geometric applications. In fact what may now seem like a simple consequence of the definition of derivative, is the solution to a problem that had vexed mathematicians for centuries: how to find the line tangent to *any* curve at a given point.

Technical fact

If the line tangent to $f(x)$ at $(c, f(c))$ exists, its equation is given by:

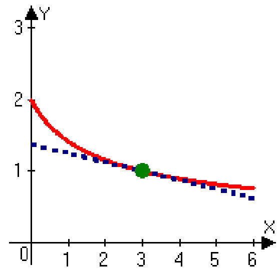
$$y = f'(c)(x-c)+f(c)$$

Proof

In this situation, we know a point on the tangent line and can compute its slope by computing the derivative. By using the point-slope formula, we obtain the stated formula.

$$y = \underset{\substack{\uparrow\\ m}}{f'(c)}(x-\underset{\substack{\uparrow\\ x_0}}{c})+\underset{\substack{\uparrow\\ y_0}}{f(c)}$$

Example: $y=\dfrac{2}{\sqrt{x+1}}$ *at* $(3,1)$



To find the equation of this tangent line we can use the given point, which is on the curve, and its slope, which we can find by using the definition of derivative:

$$m=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{\dfrac{2}{\sqrt{3+h+1}}-1}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{2}{\sqrt{4+h}}-1}{h}=\lim_{h\to 0}\frac{2-\sqrt{4+h}}{h\sqrt{4+h}}$$

By rationalizing, this becomes:

$$=\lim_{h\to 0}\frac{2-\sqrt{4+h}}{h\sqrt{4+h}}\,\frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}=\lim_{h\to 0}\frac{4-4-h}{h\sqrt{4+h}\left(2+\sqrt{4+h}\right)}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{4+h}\left(2+\sqrt{4+h}\right)}=-\frac{1}{8}$$

Therefore the required equation is:

$$y=f'(3)(x-3)+f(3)=-\frac{1}{8}(x-3)+1$$

We can change this to a slope-intercept form, but only if needed.

This use of derivatives reminds me of a joke made by a colleague and friend of mine who, while discussing a book at a conference, stated that "*This book on calculus is not very good: some chapters go off on tangents.*"

Is that supposed to be funny?

If you understand this interpretation of a derivative, it should be: all derivatives take us to tangents, don't they?

Moving right along…

OK, ☹ . Since derivatives indicate slopes, we can use them to obtain the following information about graphs of functions, which you probably remember from high school.

Knot on your finger

At any point where ***the derivative is negative***, the function is ***decreasing*** (going down) since its slope is negative.

At any point where ***the derivative is positive***, the function is ***increasing*** (going up) since its slope is positive.

We'll use this property very frequently later, when we shall focus on graphs.

We can now look at the other interpretation of a derivative: as a rate of change.

Knot on your finger

The ***practical*** interpretation of the derivative of a function $f(x)$ is as the ***instantaneous rate of change*** of the dependent variable in relation to the independent variable.

Example:

Assume that the cost of producing *x* thousands candy bars is given by the function $c(x)=0.02x^2-0.3x+125$ dollars. Although this is a very simple model, the quadratic term may be generated by labour costs, the linear term may be operational costs, while the constant term may be related to rent, electricity, etc.

In this case the derivative $c'(x)$ tells us how fast such costs increase as the production increases. As an additional exercise, you may want to use the definition (not any differentiation rules you may remember from high school) to check that this derivative is $c'(x) = 0.04x - 0.3$. Therefore, for low values of *x* this derivative is negative and the cost function is decreasing. However for larger values of *x*, the derivative becomes positive and the cost function increases. Finding the best production level, which corresponds to the lowest cost, is an interesting problem that we shall explore more later.

Remembering the geometrical and practical interpretations of the derivative we can interpret the concept of *differentiability* as follows.

Knot on your finger

If a function is ***differentiable*** at a point, then:

- It has a ***non-vertical tangent line*** at that point
- It changes at a well-defined, ***finite rate***.

The easiest and most common use of the interpretation of derivatives as a rate of change occurs in motion problems.

I will first remind you of a basic physics definition which I am sure is very familiar to you.

Definition

If $y = f(t)$ represents the ***position*** of a moving object as a function of time, then the ***average velocity*** of the object between $t = a$ and $t = b$ is given by:

$$v_{avg} = \frac{space}{time} = \frac{f(b) - f(a)}{b - a}$$

By using derivatives, we can take our understanding of velocity a little deeper.

Technical fact

If $y = f(t)$ represents the ***position*** of a moving point as a function of time, then the ***instantaneous velocity*** of the object is provided by its ***derivative***:

$$v(t) = \frac{dy}{dt} = f'(t)$$

Proof

To compute the instantaneous velocity at $t = a$, we consider *t* as a variable and let it approach *a*:

$$v_{inst} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$$

But this is the alternative definition of derivative, hence the claim.

A similar argument (that you can generate on your own as an exercise) leads to another common use of derivatives in motion problems.

Technical fact

If $v = f(t)$ describes the ***instantaneous velocity*** of a moving point as a function of time, then its ***derivative*** represents the ***instantaneous acceleration*** of the object:

$$a(t) = \frac{dv}{dt} = f'(t)$$

Example: $y = \dfrac{2}{\sqrt{t+1}}$

If the position of a moving object in time is described by this function, then its velocity at time $t = 3$ is given by the derivative, which we computed in the previous example as being $-\frac{1}{8}$. The fact that this value is negative implies that the object is moving downward.

What about the acceleration? It is the derivative of the velocity function:

$$y' = -\frac{1}{\sqrt{(t+1)^3}}$$

Since we have not seen differentiation rules yet, computing this derivative or its own derivative by using the definition can be quite laborious. Therefore, we delay such computation to later.

For now make sure that you remember the principle: the acceleration is the derivative of the velocity.

Summary

- A derivative can be interpreted as the slope of the tangent line, or as the rate of change at which the dependent variable changes with respect to the independent variable.
- In particular, the derivative of a position function indicates velocity and the derivative of a velocity function indicates acceleration.

Common errors to avoid

- Don't rush when interpreting the practical meaning of a derivative.

Learning questions for Section D 3-3

Review questions:

1. Explain why the derivative of a function represents the slope of the tangent line.
2. Explain why the derivative of a function represents the rate at which the dependent variable changes with respect to changes in the independent variable.
3. Explain why the derivative of a position function represents the velocity of the object.
4. Explain why the derivative of a velocity function represents the acceleration of the object.

Memory questions:

1. If $f(a) = k$ and $f'(a) = p$, what is the equation of the line tangent to the function $f(x)$ at $x=a$?
2. If $f'(x) < 0$, what can we say about the function $f(x)$?
3. If $f'(x) > 0$, what can we say about the function $f(x)$?
4. If the function $s = s(t)$ describes the position of a moving object, which formula describes the average velocity of the object between $t = a$ and $t = b$?
5. If the function $s = s(t)$ describes the position of a moving object, which limit formula describes the instantaneous velocity of the object at $t = a$?
6. Which physical quantity is represented by the derivative of a position function?
7. Which physical quantity is represented by the derivative of a velocity function?

Computation questions:

For each of the functions in questions 1-10, use the definition of derivative to find the slope and then the equation of the line tangent to the curve at the given point.

1. $y = \frac{2x}{\sqrt{x+1}}$ at $(0, 0)$.
2. $f(x) = x^2 - x$ at $(3, 6)$
3. $f(x) = 12x + 7x^2$ at $(-1, -5)$
4. $y = x^3 + x$ at $(-3, -30)$

5. $y = 12 + \frac{7}{x}$ at $(-1, 5)$

6. $f(x) = \sqrt{x-5}$ at $(6, 1)$

7. $f(x) = \frac{2}{\sqrt{3-x}}$ at $(1, \sqrt{2})$

8. $y = \frac{1}{x+2}$ at $\left(\frac{1}{2}, \frac{2}{5}\right)$

9. $y = \sqrt[3]{x}$ at $(8, 2)$

10. $y = \sqrt[4]{x}$ at $(1, 1)$

11. In order to estimate the slope of the line tangent to $y = e^{2x}$ at (0, 1):
    a) Compute the slopes of a few lines secant to this curve, each through (0, 1) and a suitable point near it.
    b) Use the pattern of slopes so obtained to estimate the slope of the tangent line.

Theory questions:

1. Which form of the equation of a line is most convenient when constructing the tangent to a given function at a given point?

2. If the graph of $f'(x)$ is below the *x* axis, what can we say about the function $f(x)$?

3. If the graph of $f'(x)$ is above the *x* axis, what can we say about the function $f(x)$?

4. When is $f'(a)$ the slope of the line tangent to $f(x)$ at (a, b) ?

5. If you graph a polynomial function and zoom in repeatedly on any one of its points, what shape will its graph take eventually?

6. The function $f(x) = x^{1/3}$ has a tangent line at the origin, but is not differentiable there. How is that possible?

Application questions:

In each of questions 1-8, determine the practical interpretation of the derivative of the given function. For additional practice, compute such derivative by using the definition, but be aware that in some cases this may be quite involved or even too complicated for our purposes.

1. The function $F = f(v)$ represents the force exerted by a motor as a function of the voltage applied.

2. The height of a projectile *t* seconds after it is shot straight up from the Moon's surface is given by $h(t) = 252t - 0.82t^2$.

3. The thermostat of a heating system is set so that after t hours it creates a temperature of $T(t) = 28 + 5\cos\frac{\pi t}{12}$.

4. After t years in college a student knows approximately $k(t) = 572 + 73t^{5/6}$ major concepts.

5. After working for t days at a certain assembly job, an employee can assemble $N(t) = \frac{100t}{t+9}$ components per day.

6. The number of bacteria present in a culture t hours after adding a bactericide is $b(t) = 10^6 + 10^4 t - 10^3 t^2$.

7. The average pulse rate r, in beats per minute, of a normal person depends on the person's height h in cm according to the function $r(h) = \frac{1000}{\sqrt{h}}$.

8. The concentration C of a drug in the blood of a patient t hours after it is injected is given by the function $C(t) = \frac{0.12t}{t^2 + 1.3}$.

For each of questions 9-13, the position function of a moving object is presented. Use the appropriate definitions to compute the average velocity of the object in the given time interval and the instantaneous velocity at the integer valued endpoint.

9. $s = 4t^3,\ [4, 4.5]$

10. $s = 2t^2 + 3t,\ [3, 3.2]$

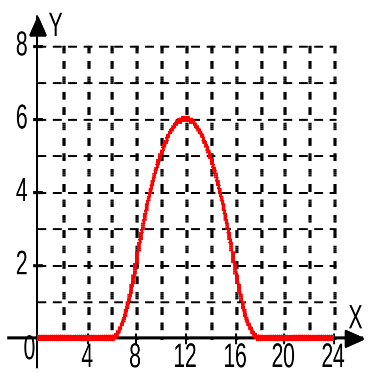
11. $y = -5t^2 + 15t + 3,\ [2.8, 3]$

12. $y = -4.9t^2 - 2t + 30,\ [1.8, 2.1]$

13. $x = \frac{2t}{t+2},\ [0, 2]$

14. $x = \frac{t}{t^2+1},\ [1, 1.2]$

15. The amount of pollutants emitted by a certain factory changes during the day according to a function whose graph is provided here. Estimate the rate of emission of this pollutant at 10 a.m. and at 4 p.m.



16. Which formula represents the initial rate of decay of a substance whose mass is given by $m = 350e^{-0.08t}$?

Templated questions:

1. Construct a simple function, pick a point on its graph and use the definition of derivative to determine the equation of the line tangent to the graph at the point.

2. In any applied problem involving a function $y = f(x)$, determine the units of $f'(x)$.

What questions do you have for your instructor?