Chapter 3: Definition of derivative — *Section 4*

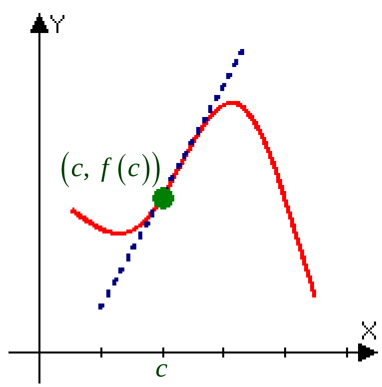
Differentials

What you need to know already:	*What you can learn here:*
➤ The definition of derivative and its interpretation in terms of slopes and rates of change.	➤ The definition of a very important technical tool in differential and integral calculus: the differential.

The purpose of this last section of the introductory chapter on derivatives is to make you acquainted with a very important concept/tool of calculus. It is a tool whose definition can seem quite mysterious at the beginning, but will become more real and concrete once you understand and become familiar with its meaning. I will start by setting the context for its meaning.

Remember that the derivative of a function $f(x)$ at a point $(c, f(c))$ is defined as the slope of the tangent line:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$



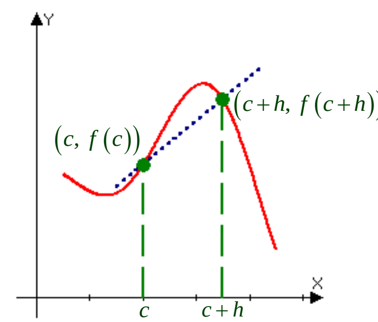
Not that formula again!

This is *THE* formula for calculus, so you should expect it all the times and become its friend, no? ☺

Now, as you hopefully remember, this formula gives us the limit of a secant slope, so that the numerator of the fraction corresponds to the rise of such line and the denominator to the run. When we compute the limit, if it exists, we obtain the slope of the tangent line.

But that means that when h is just a small value, before we go to the limit, we have a reasonable approximation of such slope, given by:

$$f'(c) \approx \frac{f(c+h) - f(c)}{h}$$



It is this approximation that we shall meet in this section and then use often in the future.

Notice what it says:

Knot on your finger

The ***derivative*** of $f(x)$ at $(c, f(c))$ is ***approximately equal*** to the ratio of rise to run between $(c, f(c))$ and some nearby point on the graph of the function.

Amazingly, to develop the tool of differentials all we need is setting the notation and terminology properly.

Definitions

In the context of the defining formula for the derivative, the small quantity $h = x - c$ is also denoted by:

- Δx, and called the ***increment*** of *x*, when viewed as a change in *x* affecting the function.
- dx and called the ***differential*** of *x*, when viewed as a change in *x* affecting the tangent line.

In both cases we are dealing with the ***run*** of the slope between *c* and $c+h$.

I don't get it: why call the same thing in three different ways?

I told you that it would look mysterious at the beginning. Bear with me.

Now we are going to play the same terminology game with the *y* variable. And this is where the substance of the definition and the concept lie.

Definitions

The quantity:

$$\Delta y = \Delta f = f\left(c+h\right) - f\left(c\right)$$

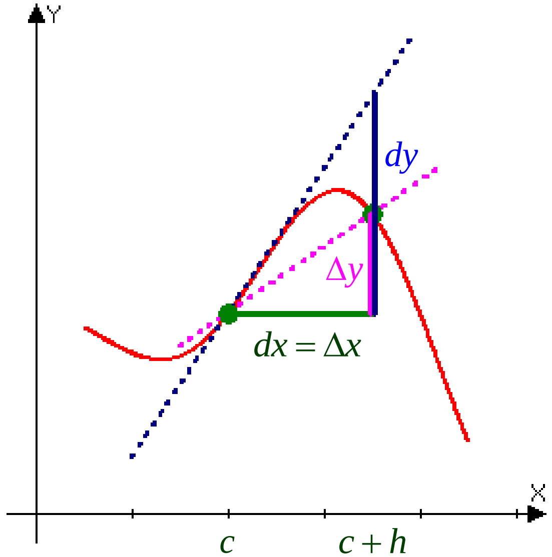
is called the ***increment*** of *y* and it represents the rise in the slope of the ***secant*** line.

Moreover, the quantity:

$$dy = df = f'(c)\, dx$$

is called the ***differential*** of *y* at *c* and it represents the rise portion of the slope of the ***tangent*** line.

I hope that this combined (and larger) picture will help you understand the meaning of all these pieces.



Notice that $h = dx = \Delta x$ simply tells us how far we have moved from *c*. And remember that whatever it is, it's supposed to be a small value, since we are not moving far from *c*.

Moreover, Δy tells us how much the value of the function has changed in going from $\left(c,\ f\left(c\right)\right)$ to $\left(c+h,\ f\left(c+h\right)\right)$: it is the ***rise along the curve***.

Finally, *dy* tells us how much the value of function representing the tangent line has changed in going from $x = c$ to $x = c+h$: it is the ***rise along the tangent line***.

And what's the point?

Good question! The point is that as long as h is small, Δy and dy are close to each other, so they can be used to approximate each other. Since:

$$f'(c) \approx \frac{f(c+h)-f(c)}{h}$$

we can multiply both sides by h to obtain:

$$f'(c)\times h \approx f(c+h)-f(c)$$

By using our new notation, this implies that:

$$\Delta y = f(c+h)-f(c) \approx f'(c)\times \Delta x = f'(c)\times dx = dy$$

Ok, if you say so.

I realize that this may still look mysterious and I am not asking you to make full sense of it as yet. For now, just learn the terminology and notation and be prepared to be amazed by what these differentials will allow us to do in the future.

So, for now, just be ready to perform the kind of work shown in this example.

Example: $f(x)=\dfrac{3}{x^2}$

Let us first compute the increment of this function for $x=2$ and $\Delta x = 0.1$.

This means that we need to compute:

$$f(x+h)-f(x)=f(2.1)-f(2)=3\left(\frac{1}{2.1^2}-\frac{1}{2^2}\right)\approx -0.0697$$

Now, let us compute its differential. In section 3-2 we saw that the derivative of this function is:

$$f'(x)=-\frac{6}{x^3}$$

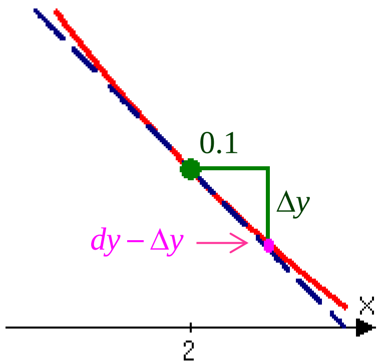
Therefore the differential is:

$$dy = f'(x)dx=\left(-\frac{6}{x^3}\right)dx$$

same; they approximate, but do not replace or equate each other.

But when I look back at that large picture you had, the increment and the differential do not look close at all!

That is because the picture was exaggerated to show each piece clearly. In that way the differential of x was not small enough to generate a good approximation. In the uses we shall make, the differential of x will be much smaller, closer to 0, and things will work out wonderfully. Look at the corresponding picture for this last example:



Can you see the difference between dy and Δy even in this enlarged scale?

Hmmm, I see! I guess I have to trust you for now.

Yes, you have been trusting me on so many levels, add this one more item: you will not be disappointed!

Summary

- The increments Δx and Δy of a function at a point represent the small amount by which the independent and dependent variable, respectively, change close to a given point.
- The differential dy of a function is simply given by the product $dy = f'(x)\,dx$ and its many uses will be seen in the rest of the course.

Common errors to avoid

- At this stage, and until you see more uses of differentials, don't read too much into them. Instead, focus on the definition and how to implement it in simple cases.

Learning questions for Section D 3-4

Review questions:

1. Try to describe what a differential is and what it represents geometrically, both for the *x* and the *y* variables.

Memory questions:

1. What does the symbol Δx represent?
2. What does the symbol Δy represent?
3. Which formula defines the increment Δy of a function $f(x)$ at $x = c$?
4. Which formula defines the differential dy of a function $f(x)$?
5. What are the three common symbols used to identify the differential of the independent variable?

Computation questions:

In each of questions 1-10, determine all the increments and differentials from the given information. In particular, compare the values of Δy and dy. Although the derivative function is given for your convenience, make sure you know how to obtain it by using only the definition of derivative.

1. $f(x) = x^2 - x$ at $(3, 6)$ with $\Delta x = 0.1$. $f'(x) = 2x - 1$

2. $f(x) = 12x + 7x^2$ at $(-1, -5)$ with $\Delta x = 0.2$. $f'(x) = 12 + 14x$

3. $y = \dfrac{1}{x+2}$ at $\left(\dfrac{1}{2}, \dfrac{2}{5}\right)$ with $\Delta x = 0.25$. $y' = -\dfrac{1}{(x+2)^2}$

4. $y = 12 + \dfrac{7}{x}$ at $(-1, 5)$ with $\Delta x = 0.2$. $y' = -\dfrac{7}{x^2}$

5. $y = \dfrac{1}{1+x}$ at $x = 0$ with $dx = \dfrac{1}{36}$. $y' = -\dfrac{1}{(1+x)^2}$

6. $f(x) = \dfrac{2}{\sqrt{3-x}}$ at $(1, \sqrt{2})$ with $\Delta x = 0.3$. $f'(x) = \dfrac{1}{\sqrt{(3-x)^3}}$

7. $y = \dfrac{2x}{\sqrt{x+1}}$ at $(3, 3)$, with $\Delta x = 0.2$. $y' = \dfrac{x+2}{\sqrt{(x+1)^3}}$

8. $f(x) = \sqrt{x-5}$ at $(6, 1)$ with $\Delta x = 0.1$. $f'(x) = \dfrac{1}{2\sqrt{x-5}}$

9. $f(x) = \sqrt{45 + x^2}$ at $x = 2$ with $dx = 1.41$. $f'(x) = \dfrac{x}{\sqrt{45 + x^2}}$

10. $y = \sqrt[3]{x}$ at $(8, 2)$ with $\Delta x = 0.01$. $y' = \dfrac{1}{3\sqrt[3]{x^2}}$

Theory questions:

1. A differential dy represents the rise along what?

2. An increment Δy represents the rise along what?

3. Why are the increment and differential of the independent variable defined in the same way?

4. We have seen that $dx = \Delta x$, but for which functions is $dy = \Delta y$?

5. Where in the formula that defines a derivative is the increment of the function found?

Templated questions:

1. Compute all increments and differential for any simple function of your choice at any of it points and a small value of *h*.

What questions do you have for your instructor?