Chapter 4: Basic differentiation rules — **Section 2**

Five easy differentiation rules

What you need to know already:	*What you can learn here:*
➤ What a derivative and a differentiation rule are.	➤ How to quickly compute derivatives in five particularly easy situations.

We begin the exploration of differentiation rules by looking at five rules whose algebraic proofs are very easy and whose geometrical interpretations are also easy to understand.

We start with the simplest and, arguably, most often used rule.

Technical fact:

The constant rule

If c is a real number and $f(x) = c$, then $f'(x) = 0$

Proof

Algebraically we have that:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

Geometrically, in this case the function is a straight horizontal line, so that its slope is 0.

The constant rule can be seen as a special case of the next rule:

Technical fact:

The linear rule

If m and b are real numbers and $f(x) = mx + b$, then $f'(x) = m$.

Proof

Geometrically, we know that a function of this type is a line with slope *m*, so that the derivative, being its slope, is *m* as well.

Algebraically we have:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[m(x+h)+b] - [mx+b]}{h} =$$

$$= \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m$$

The next rule is not much more complicated.

Technical fact:

The coefficient rule

If k is a real number and $f(x) = k \cdot g(x)$, then $f'(x) = k \cdot g'(x)$

Proof

Geometrically, the coefficient is making the function k times bigger, thus making it grow k times faster. Not convinced? Let us look at the definition, as we always should to prove any differentiation rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{k\left[g(x+h)\right]-k\left[g(x)\right]}{h} =$$

$$= k\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = k\,g'(x)$$

Now we know for sure!

The next rule is deceptively simple, since its statement is easy to understand and easy to implement. HOWEVER, it is here that many students first make the mistake discussed in Section 4-1 and they do so in two ways: they think that all other rules are just as simple and they forget that there is a rule and therefore there is a proof behind it!

Technical fact:

The addition rule

If $y = f(x) + g(x)$, then $y' = f'(x) + g'(x)$.

Proof

If you think that this fact is obvious, think again: we need a proof ! It's an easy proof, but we need it.

We start, as usual, from the definition:

$$\frac{d}{dx}\left(f(x)+g(x)\right) = \lim_{h\to 0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

We now switch the two middle terms and split the fraction:

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} =$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)$$

Finally we use the anti-Murphy's law, split the limit and recognize each term as the definition of the corresponding derivative:

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$$

As claimed!

Does this work for subtraction too?

Yes, in fact, the subtraction rule is really just a special case of the addition rule and does not count as one of the five. I leave to you the fun of proving it in one of the *Learning Questions*.

The last easy rule may come as a bit of a surprise, since it refers to one of the transcendental functions with which you are less familiar, but both the rule and its proof are indeed easy.

Technical fact:

The natural exponential rule

The derivative of $f(x)=e^x$ is $f'(x)=e^x$

Proof

As usual, we go back to the definition:

$$\frac{d}{dx}f(x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}=$$

$$=e^x\lim_{h\to 0}\frac{e^h-1}{h}=e^x f'(0)$$

But now we remember that the number e was defined as the base of that special exponential function whose slope at $x=0$ is 1. Therefore:

$$f'(0)=1 \quad\Rightarrow\quad f'(x)=e^x$$

That should be easy to remember!

It usually is. Now, before I show you an example, an important point for you to remember.

Strategy for applying differentiation rules

All differentiation rules can be ***combined*** to compute the derivative of a function, provided that each of them is applied ***appropriately*** and in the way that its "*if*" clause requires.

This may require ***rearranging*** the function in a different way in order to properly use each of the rules.

The rearrangement mentioned in the strategy is part of a general ***get ready principle*** that ends up being useful in many mathematical methods.

Example: $\frac{d}{dx}\left(3x+5-5e^x\right)$

We first apply the addition rule:

$$\frac{d}{dx}\left(3x+5-5e^x\right)=\left(3x+5\right)'-\left(5e^x\right)'$$

For the first term we can use the linear rule, while on the second we can use the coefficient rule:

$$\frac{d}{dx}\left(3x+5-5e^x\right)=3-\left(5e^x\right)'=3-5\left(e^x\right)'$$

Finally we use the natural exponential rule to arrive at the conclusion:

$$\frac{d}{dx}\left(3x+5-5e^x\right)=3-5e^x$$

All done with no limits: that's the point of differentiation rules!

Can we get more examples?

Yes, but with the five rules we have seen here, we cannot go very far in computing derivatives, so any other example would be a variant of the one above.

But fear not, more complicated cases and examples will arrive soon.

Summary

- The constant, linear, coefficient, addition and natural exponential rules are all very simple and easy to prove, but they are rules and must be followed.

Common errors to avoid

- Do not use a rule in a situation in which it does not apply!

Learning questions for Section D 4-2

Review questions:

1. Explain why the constant and linear rules work.
2. Identify what is in common among the proofs of the five rules presented in this section.

Memory questions:

1. Which formula describes the constant rule?
2. Which formula describes the linear rule?
3. Which formula describes the coefficient rule?
4. Which formula describes the addition rule?
5. Which formula describes the natural exponential rule?

Computation questions:

1. Compute the derivative of the function $y = ax - b - ce^x$ by using the five rules of this section and identify each rule at the step where it is used.

2. Determine the value of *m* and *b* for which the function $f(x) = \begin{cases} mx + b & if\ \ x \le 3 \\ mx^2 + \frac{b}{3}x + 2 & if\ \ x > 3 \end{cases}$ is differentiable everywhere.

Theory questions:

1. Which of the differentiation rules provides the definition of a derivative?

2. What is the geometrical reason behind the constant rule?

3. Why are the constant rule and the linear rule correct?

4. So, why is *e* used as the preferred base for exponential functions?

5. To which function(s) does the natural exponential rule apply?

6. Can the derivative of the function $y = e^{-x}$ be obtained by using the natural exponential rule only?

7. Identify one function that is equal to its derivative.

8. Explain why all possible computation questions for this section would look like the first computation question, which is equivalent to the templated question below.

3. Some books suggest a *linearity rule*, which says that $\left(c\ f(x) + g(x)\right)' = c\ f'(x) + g'(x)$. Which rules of this section does this one combine?

Proof questions:

1. Provide two different proofs of the fact that if $y = f(x) - g(x)$, then $y' = f'(x) - g'(x)$.

2. Prove that the *x*-intercept of the line tangent to $y = e^x$ at $\left(c, e^c\right)$ is at $x = c - 1$ for any value of *c*. Then use this fact to determine the length of the segment that joins the point and the intercept.

Templated questions:

1. Compute the derivative of any function of the form $f\left(x\right) = ax + b + ke^x$ and identify each of the differentiation rules you use and at which step you use them.

2. When computing the derivative of a given function, by using differentiation rules, identify each and every rule you use, in the specific order in which you use it. If a rule is used more than once, identify each instance where it is used.

What questions do you have for your instructor?