Chapter 4: Basic differentiation rules *Section 3*

Product, quotient and power rules

What you need to know already:	*What you can learn here:*
➤ The definition of derivative and the most basic differentiation rules.	➤ How to compute the derivative of a function that consists of a product, quotient or power.

The basic rules described in the last section only allow us to compute the derivative of very simple functions. What do we do when the function is defined by a function that cannot be rewritten as a sum of linear and exponential pieces? In other words, how do we compute the derivative of a product, quotient, power or a combination of them?

The answer is fairly simple, but not intuitive, as the case of the product will show. In fact, even Leibniz made the initial mistake of believing that, in analogy to what happens in the addition rule, the derivative of a product could be obtained as the product of the derivatives. He quickly realized his error, but don't follow him in such error! Here is the proper way to compute the derivative of a product.

Technical fact:

The product rule

If $y = f(x)\times g(x)$, then:

$$y' = f'(x)\,g(x) + f(x)\,g'(x)$$

Why do we have to make it so complicated?

We don't make it complicated: this is the way it is! Whether we like it or not, this is the proper way to compute the derivative of a product (that's why we call it a *rule*) and here is why.

Proof

By the definition of derivative (what we must always return to):

$$(f\,g)'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We now use the common mathematical method of adding and subtracting a useful quantity

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

We now split the fraction into two more familiar pieces

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}\right)$$

We can now compute the limit of each piece separately, first by factoring the common factor that exists in each fraction:

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}g(x+h) + \lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h}$$

and then computing the limit of each piece. Our "*anti-Murphy's law*", the definition of the derivative for $f(x)$ and $g(x)$ separately, as well as the fact that differentiable functions are continuous, imply that:

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\times\lim_{h\to 0} g(x+h) + f(x)\times\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$= f'(x) \qquad g(x) \qquad + f(x) \qquad g'(x)$$

That is:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)\,g(x) + f(x)\,g'(x)$$

And we have to use this long formula every time we need to differentiate a product?

Only if you want to get the *correct* formula! If you don't care for correctness, you can use whatever you want. But in that case, you should not waste your time getting an education, eh? ☺

Jokes apart, I see your point; this formula is not intuitive, but it is the correct one and we want to do things right, so we must use it.

Example: $f(x) = xe^x$

Since this function is neither linear, nor exponential, but the product of the two, we need the product rule. Therefore:

$$f'(x) = (x)' e^x + x\left(e^x\right)'$$

Now we can apply the linear and exponential rules to the appropriate pieces, to obtain:

$$f'(x) = 1 \bullet e^x + xe^x = e^x(1+x)$$

By the way, notice that the formula for the product rule is symmetric, since the product also is symmetric. So:

Knot on your finger

When applying the product rule, the following formulae are also ***acceptable***:

$$(f\,g)' = g(x)f'(x) + g'(x)f(x)$$

$$(f\,g)' = g'(x)f(x) + f'(x)g(x)$$

Feel free to use whichever option is easiest for you to remember; only, do not use a different, *incorrect* formula!

Do we need a similar tricky proof to construct a quotient rule?

If by "*tricky*" you mean one that uses the method of adding and subtracting, no. But we shall use a different and equally nice *trick*.

Technical fact:

The quotient rule

If $y = \dfrac{f(x)}{g(x)}$, then $y' = \dfrac{f'\,g - f\,g'}{g^2}$

Proof

Instead of using the adding and subtracting method, we use two other important mathematical methods. First, to keep the notation simple, we omit the x in our formulae, but remember that in what follows y, f, g represent functions of x.

Next, we use the golden rule of equations: "*do to the left side what you do to the right side*." So, we begin by multiplying both sides by *g*:

$$y=\frac{f}{g} \quad \Rightarrow \quad y\times g=f$$

Now we compute the derivatives on both sides, using the product rule on the left:

$$(yg)'=f' \quad \Rightarrow \quad y'g+yg'=f'$$

Now we remember that we are looking for *y'* and we know what *y* is. Therefore:

$$y'g+yg'=f' \quad \Rightarrow \quad y'g=f'-yg'=f'-\frac{f}{g}g'$$

$$\Rightarrow \quad y'=\frac{f'}{g}-\frac{fg'}{g^2}=\frac{f'\,g-f\,g'}{g^2}$$

And this is the quotient rule formula.

Example: $y=\dfrac{e^x}{x}$

By applying the quotient rule we get:

$$y'=\frac{(e^x)'\,x-e^x\,(x)'}{x^2}$$

We now use the linear and exponential rules to conclude that:

$$y'=\frac{xe^x-e^x}{x^2}=e^x\,\frac{x-1}{x^2}$$

Notice that since the quotient is not a symmetric operation $(2/3\neq 3/2)$, in the quotient rule formula we switch terms, or factors or take reciprocals in a haphazard way.

Example: $y=\dfrac{x}{e^x}$

Although this function is the reciprocal of the one in the last example, we cannot obtain its derivative by using the reciprocal of the previous derivative! We can, however, apply the quotient rule in the proper way, thus getting:

$$y'=\frac{e^x\,(x)'-(e^x)'\,x}{(e^x)^2}=\frac{e^x-e^x x}{(e^x)^2}=\frac{1-x}{e^x}$$

This is not the reciprocal of the earlier derivative, nor can it be related to it in any non-coincidental way. Compute the two separately!

The title of this section includes mention of the power rule: is that because powers are products?

Certainly! In fact, here is how we can compute the derivative of $f(x)=x^2$.

Example: $y=x^2$

Since we can write $x^2=x\bullet x$, we can use the product rule:

$$(x^2)'=(x\,x)'=(x)'\,x+x\,(x)'=x+x=2x$$

But you probably know that this is just a special case of a general rule, so here is the rule.

Technical fact: The power rule

If $y = x^c$, where c is any real number, then:

$$y' = c\,x^{c-1}$$

Right! That's the derivative formula!

Careful! Many people (mostly those who don't really know calculus) consider this to be the definition of derivative, while we know that it is NOT! This formula only works for power functions. Moreover, although it is a famous and simple formula, its proof is not that easy. You can certainly use it from now on, but all I can offer for now is a proof in the case where the exponent is an integer. You will see later a proof for when the exponent is a fraction, but I will not show you a proof for when it is an irrational number. Such a proof exists, but it is rather technical and beyond our goals. If you are interested, look it up on a suitable resource.

So, here is what I can offer for now.

Partial proof

We have already seen that this formula works for $c = 1$ (the linear rule) and for $c = 2$ (last example). So, let us assume that it works when c is a positive integer up to, say, $c = n$. Notice that we know that it works up to 2.

In that case we can use the product rule again to show that it works for $c = n+1$:

$$\left(x^{n+1}\right)' = \left(x \bullet x^n\right)' = (x)'\left(x^n\right) + x\left(x^n\right)' = x^n + x\left(nx^{n-1}\right)$$

But this expression simplifies to:

$$x^n + x\left(nx^{n-1}\right) = x^n + nx^n = (n+1)x^n$$

This is exactly what we get from our formula for $c = n+1$. By this thinking, the formula must work for every positive integer, since every time we find that it works for one, it must work for the next one too!

By the way, this is an example of the method of *proof by induction*.

Now that we know it works for positive integers, we can show that it works for negative integers too, by using the quotient rule:

$$\left(x^{-n}\right)' = \left(\frac{1}{x^n}\right)' = \frac{(1)'\,x^n - 1\left(x^n\right)'}{x^{2n}} = \frac{0\bullet x^n - nx^{n-1}}{x^{2n}} = (-n)\,x^{-n-1}$$

And we have our formula again.

I have not shown that the formula works when $c = 0$, but I will leave this easy check to you.

Can we see an example?

I am offering you one massive example that includes all the rules we have seen so far. The *Learning questions* and any other example you can come up with on your own will provide as much fun and entertainment… I mean as much practice as you want. ☺

Example: $\dfrac{d}{dx}\left(\dfrac{x^3e^x}{5+x+2x^{-3}}\right)$

Since the whole function is given by a quotient we begin with the quotient rule:

$$\frac{d}{dx}\left(\frac{x^3e^x}{5+x+2x^{-3}}\right) = \frac{\left(x^3e^x\right)'\left(5+x+2x^{-3}\right) - \left(x^3e^x\right)\left(5+x+2x^{-3}\right)'}{\left(5+x+2x^{-3}\right)^2}$$

For the first remaining derivative, we use the product rule, followed by power and exponential rule to obtain:

$$= \frac{\left(3x^2e^x + x^3e^x\right)\left(5+x+2x^{-3}\right) - \left(x^3e^x\right)\left(5+x+2x^{-3}\right)'}{\left(5+x+2x^{-3}\right)^2}$$

For the second and last remaining derivative, we use the linear (which includes constant and addition rules), coefficient and power rule again:

$$= \frac{\left(3x^2e^x + x^3e^x\right)\left(5 + x + 2x^{-3}\right) - \left(x^3e^x\right)\left(1 - 6x^{-4}\right)}{\left(5 + x + 2x^{-3}\right)^2}$$

Are you missing something?

What?

No simplifying?

Why? All we needed to do was compute the derivative. We have done that, so the simplest thing to do is leave it as is! In later sections, when we shall need to use the derivative for something else, we shall change it appropriately, but now there is no need to waste our time on useless manipulations.

I like that!

I am glad, so, on that note, go on to your learning activities.

Summary

- The product, quotient and power rules are very useful and key differentiation rules: become very familiar with them!
- Each of these rules has a proof behind them that confirms their validity: become familiar with them too!

Common errors to avoid

- Follow the proper rules and don't make up your own, unless you can prove that they are correct!

Learning questions for Section D 4-3

Review questions:

1. Explain the method used to prove the product rule.
2. Explain the method used to prove the quotient rule.
3. Explain the method used to prove the power rule.
4. Explain how the mnemonics "*One-D-two plus two-D-one, calculus is so much fun*" and "*Two-D-one plus one-D-two, calculus can be fun too*" can help you remember the product rule.
5. Explain how the mnemonic "Low-D-high minus high-D-low, draw a line and square below can help you remember the quotient rule.

Memory questions:

1. Which formula describes the product rule?
2. Which formula describes the quotient rule?
3. Which formula describes the power rule?
4. For which values of the exponent is the power rule valid?

Computation questions:

For each of the functions presented in questions 1-12, use appropriate algebraic manipulation and appropriate differentiation rules from this section to compute its derivative. In each case identify which rule is used at which step.

1. $y = \dfrac{3}{4x^7}$
2. $y = \sqrt[5]{x^3} - e^{x+5}$
3. $y = \dfrac{3x^3 + x^2 - 5}{x}$
4. $y = e^x x^4$
5. $y = \dfrac{e^x + 1}{e^x - 1}$
6. $y = e^{-x}$
7. $y = \dfrac{\sqrt{x} - e^x}{x^3 + 5}$
8. $y = \dfrac{\sqrt{x}e^x}{3x^4 + 8x}$
9. $y = (2x-5)^2$
10. $y = (3x-2)^3$
11. $y = \dfrac{\sqrt{x^3} - 3x^5}{x^3 - 5x}$
12. $y = \dfrac{\sqrt{x} + \dfrac{1}{\sqrt{x}}}{3x^4\left(\sqrt{x}+5\right)}$

13. Compute the derivative of the function $y = \left(\sqrt{x} + 5x^2 - x\right)\left(\sqrt{x} - \dfrac{1}{\sqrt{x}} + 2\right)$ by using two substantially different sequences of differentiation rules.

Theory questions:

1. Which basic mathematical *trick* is used to prove the product rule?

2. The proof of which basic differentiation rule uses the *"add and subtract"* method?

3. Give an example of a function that is written as a product, but for which the product rule is not needed in differentiation.

4. If $f(x)$ and $g(x)$ are two differentiable functions, is $y = f(x)g(x)$ always differentiable?

Proof questions:

1. Check that the "*intuitive*" formula $(fg)' = f'\,g'$ is wrong by showing that this formula gives the wrong answer for the derivative of $y = x^5 = x^2 x^3$, while the correct product rule formula gives the proper answer.

2. Prove that the power rule works also for $c = 0$

3. Check that the power rule works for $y = x^3$ by using the definition of derivative only.

4. Prove that if $f(x) = ax^3 + bx^2 + cx + d$ is any cubic function with *x* intercepts at $(h, 0), (k, 0), (j, 0)$, the tangent line to the same cubic at $x = \dfrac{h+k}{2}$ intersects the cubic itself at $(j, 0)$.

5. Use appropriate differentiation rules to compute the derivative of the function $f(x) = \dfrac{ax^2 - bx}{cx - dx^2}$.

6. We have seen in section D3-4, that at any value *c* for which a function $y = f(x)$ is differentiable, $dy = f'(x)dx$, where the differential *dy* on the left side can be interpreted as the rise along the tangent line corresponding to the run *dx* at the point $(x, f(x))$.

   Now, consider the function $y = x^2$ and a point (x, x^2) on it and consider another point nearby, say (z, z^2). The rise between these two points is $\Delta y = z^2 - x^2$, which, by factoring, can be written as $z^2 - x^2 = (z + x)(z - x)$. But $z - x = \Delta x$, so that we can write $\Delta y = (z - x)\Delta x$. Since this equation is valid for any value of *z*, it is valid also when *z* is very close to *x*. In the limit, the equation becomes $dy = (x + x)dx = 2xdx$. But, by the definition of differential, this means that $f'(x) = 2x$, as the power rule states also! Is this a fluky coincidence?

   No, it isn't! In fact it provides a method to compute derivatives that works whenever the factoring algebra is possible to implement. So, prove that the same method can be used to show that:

   a) $\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$ b) $\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}$

Application questions:

1. Show that the cubic $y = x^3 - x^2 - 4x$ never has a slope of -5.

2. Show that the function $f(x) = \dfrac{xe^x}{x+1}$ never has a horizontal tangent line.

3. Find the equation of both lines tangent to the function $y = x^2$ and containing the point $(3, 2)$.

4. The frequency of vibration of a violin string is given by the function $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}}$, where L is the length, T is the tension and ρ is the linear density of the string. Determine $\dfrac{df}{d\rho}$ and explain what it represents.

Templated questions:

1. Construct a simple – but not TOO simple – function whose derivative can be computed with the rules we have seen so far and then compute it.

2. When computing the derivative of a given function, by using differentiation rules, identify each and every rule you use, in the specific order in which you use it. If a rule is used more than once, identify each instance where it is used.

What questions do you have for your instructor?