Higher order derivatives

What you need to know already:

- Basic differentiation rules.

What you can learn here:

- How to repeat the process of differentiation to obtain derivatives of derivatives, that is, higher derivatives.

Once we have computed the derivative of a function $y = f(x)$, we end up with another function $y = f'(x)$. Why not take the derivative of this new function, and then *its* derivative and so on?

Because we have better things to do with our time!

But what if that repetition provides something useful? And it does, so let us set the notation and terminology for this simple concept.

Definition

The derivative of the derivative of a function $f(x)$ is called the ***second derivative*** of that function and is denoted by one of the symbols:

$$f''(x) \qquad y'' \qquad \frac{d^2y}{dx^2}$$

Why is the exponent of 2 placed differently on the numerator and denominator?

There is a reason for that, but it is linked to certain operations that are done in advanced calculus. For now, think of it just as a strange quirk with which you need to live! And being ambitious, we do not stop here.

Definition

The derivative of the second derivative of a function $f(x)$ is called the ***third derivative*** of that function and is denoted by one of the symbols:

$$f'''(x) \qquad y''' \qquad \frac{d^3y}{dx^3}$$

Now the game can be continued, so as to define the fourth, fifth, …, *n*-th derivative of a function, for any integer number *n*.

Definition

Repeating the process of differentiation *n* times generates the ***n-th derivative*** of a function, which is denoted by one of the symbols:

$$f^{(n)}(x) \qquad y^{(n)} \qquad \frac{d^n y}{dx^n}$$

If $n \geq 2$, then the *n*-th derivative is also said to be a ***higher derivative of order n***.

Now you have changed the notation again: numbers in brackets?

I'd love to take credit for this notation, but it is one that has been used universally for a long time. After the third derivative, we would need to insert many prime symbols and they would be difficult to count. Adopting the notation of exponents in brackets keeps things easier.

OK, but what is all this stuff useful for, besides being a perversely fun game?

It turns out that these *higher* derivatives have many concrete meanings in applications. Remember that a derivative indicates a rate of change, so each higher derivative represents the rate of change of the previous one. The most common and famous applications of this idea are for graphs and motion functions.

Technical fact

The second derivative of a function $y = f(x)$ represents the rate at which the slope changes. This is called the ***concavity*** and we shall analyze it later in more detail.

Technical fact

If $x = x(t)$ is a function of time that describes the position of a moving object, then:

1. The first derivative $x'(t)$ represents the ***velocity*** of the object.
2. The second derivative $x''(t)$ represents the ***acceleration*** of the object.
3. The third derivative $x'''(t)$ represents the ***jerk*** of the object.

Proof

Just think of the meaning: the second derivative tells us how fast the velocity changes and that is acceleration. The third derivative tells us how fast the acceleration changes and that tells us how jerky the motion is.

How to use even higher derivatives will become apparent in the study of infinite series.

I can wait! For now, can we see how the game is played?

Example: $f(x) = \dfrac{x^2 + 2}{x - 1}$

To compute the first derivative of this function, we begin with the quotient rule, followed by addition, power and constant rules for the numerator, and linear for the denominator:

$$f'(x)=\frac{\left(x^2+2\right)'(x-1)-(x-1)'\left(x^2+2\right)}{(x-1)^2}$$
$$=\frac{2x(x-1)-\left(x^2+2\right)}{(x-1)^2}$$

Before computing the second derivative, we better rearrange the numerator to make our job easier:

$$f'(x)=\frac{x^2-2x-2}{(x-1)^2}$$

Now we use the same procedure (quotient rule followed by others for each bracket) to get the second derivative:

$$f''(x)=\frac{\left(x^2-2x-2\right)'(x-1)^2-\left[(x-1)^2\right]'\left(x^2-2x-2\right)}{(x-1)^4}$$
$$=\frac{(2x-2)(x-1)^2-2(x-1)\left(x^2-2x-2\right)}{(x-1)^4}$$

Rearrangements of this fraction are possible and easy, but not required, unless we need to compute the third derivative, something that you may want to do as a further exercise for you.

Example: $f(x)=(2x+3)^5$

By using the chain, power and linear rules we can compute the first few derivatives of this function easily:

$$f'(x)=5(2x+3)^4\times 2$$

You may be tempted to combine the coefficients, but experience tells us not to do that yet, and here is why. Let us look at the next few derivatives:

$$f''(x)=5\times 4(2x+3)^3\times 2\times 2=5\times 4(2x+3)^3\times 2^2$$
$$f'''(x)=5\times 4\times 3(2x+3)^2\times 2^2\times 2=5\times 4\times 3(2x+3)^2\times 2^3$$

Therefore, by keeping the coefficients separate we can identify a pattern for each group of factors that may simplify the notation. We keep going:

$$f^{(4)}(x)=5\times 4\times 3\times 2(2x+3)\times 2^4$$
$$f^{(5)}(x)=5\times 4\times 3\times 2\times 2^5$$
$$f^{(n)}(x)=0 \quad if \quad n>5$$

Remember that $f^{(4)}(x)$ and $f^{(5)}(x)$ are called higher derivatives of order 4 and 5 respectively.

The product of successive integers that you see in this example is something that often appears when computing higher derivatives. Mathematicians have agreed on a special symbol for that, one that you may have seen before.

Definition

For any positive integer *n*, the product of the first *n* integers is denoted by $n!$:

$$n!=1\times 2\times 3\times\cdots\times n$$

and it is called the ***factorial*** of *n*.

For a number of reasons, both theoretical and practical, we assume that, counter to intuition:

$$0!=1$$

I can see the use of the notation, but to say that 0!=1 makes no sense!

I may agree, except for the fact that it does! It turns out – and it is not difficult to check – that the number of ways of placing *n* objects in order is *n*! Well, how many ways do we have of ordering NO objects? Just one, right? But if this seems like a twist of logic (to see that not having anything to order is one way to order), here is another reason related to higher derivatives.

Example: $f(x)=\dfrac{1}{3x-1}$

In order to compute the derivatives of this function, it is convenient to write it as a negative power, so as to avoid the quotient rule.

$$f(x)=(3x-1)^{-1}$$

Now let us compute its higher derivatives and keep the coefficients separate, as we did before:

$$f'(x)=-1(3x-1)^{-2}\times 3$$
$$f''(x)=(-1)\times(-2)(3x-1)^{-3}\times 3^2$$
$$f'''(x)=(-1)(-2)(-3)(3x-1)^{-4}\times 3^3$$
$$f^{(4)}(x)=(-1)(-2)(-3)(-4)(3x-1)^{-5}\times 3^4$$

Notice that we have a pattern developing, since there are:

- As many negatives as the order of the derivative
- A factorial equal to the order of the derivative
- An exponent given by the negative of the order minus 1
- A power of 3 equal to the order of the derivative.

This pattern will continue, so that:

$$f^{(n)}(x)=(-1)^n(n!)(3x-1)^{-n-1}\times 3^n$$

Notice that if we let $n=0$ and assume 0!=1, this formula becomes:

$$f^{(0)}(x)=(-1)^0(0!)(3x-1)^{-0-1}\times 3^0=(3x-1)^{-1}$$

So the "0-th" derivative is the original function, as it should be!

You will see more examples of this method in the future. For now, it is time to practice on the basics you have seen here.

Summary

- Higher derivatives are obtained by successively computing the derivative of a lower order derivative.
- The *order* of a derivative refers to how many times differentiation has been performed, starting from the original function.
- For simple functions, higher order derivatives may develop a pattern that can be summarized in a single formula, often including a factorial number.

Common errors to avoid

- When looking for a pattern for the higher derivatives of a function, don't stop too soon: you may need at least 5-6 derivatives before it becomes clear.

Learning questions for Section D 4-7

Review questions:

1. Describe what a higher derivative is.

2. Describe the notation used for higher derivatives.

Memory questions:

1. Present two correct notations for the second derivative of a function $y = f(x)$

2. Present two correct notations for the *n*-th derivative of a function $y = f(x)$

3. What is the physical meaning of the second derivative of a position function?

4. What information does the second derivative contain about the graph of a function?

5. Which formula represents *n*! as a product?

6. What is the value of 0! ?

Computation questions:

In questions 1-22, compute the second and, if not too mind-boggling, third derivative of the given function.

1. $y = 12x + 7x^2$

2. $y = 12 + \frac{7}{x}$

3. $y = \frac{3}{4x^7}$

4. $y = \frac{3x}{4 + x^7}$

5. $y = \frac{3x^3 + x^2 - 5}{x}$

6. $y = \frac{3x + x^2 - x^3}{x^2}$

7. $y = \frac{2}{\sqrt{3 - x}}$

8. $y = \frac{2x}{\sqrt{x + 1}}$

9. $y = \frac{3x}{\sqrt{x - 4}}$

10. $y=\left(x^3-5x^{1/3}\right)^3$

11. $y=\left(3x^3-5\sqrt{x}\right)^4$

12. $y=4e^{x^2-x}$

13. $y=e^{-x}\sqrt{e^x+x}$

14. $y=\sqrt{\dfrac{5x-5}{3x^2-6}}$

15. $y=\sqrt[6]{x^3+x}$

16. $y=(x+1)^3\sqrt{3x-5}$

17. $y=\sqrt[5]{x^3}-e^{x+5}$

18. $y=\sqrt[5]{x^3+3x}-e^{x^2+5}$

19. $y=\dfrac{e^{\sqrt{x}}}{\sqrt{e^x}}$

20. $y=\dfrac{e^x}{\sqrt{e^{5x}}}$

21. $y=e^x x^4$

22. $y=\dfrac{e^x+1}{e^x-1}$

In questions 23-29, determine the general formula for $f^{(n)}(x)$.

23. $f(x)=\ln x$

24. $f(x)=\dfrac{1}{x+2}$

25. $f(x)=\sqrt{x-5}$

26. $f(x)=\sqrt[3]{x}$

27. $f(x)=(2x-5)^2$

28. $f(x)=e^{-x}$

29. $f(x)=e^{\frac{x}{2}}$

Theory questions:

1. What is the 100th derivative of $y=x^{50}$?

2. Which functions have a second derivative equal to 0?

3. What does the 4th derivative of a function tell us about its 3rd derivative?

4. If the product rule is needed to compute the derivative of a function, which other rule will be needed to compute the second derivative?

5. If the first derivative of a function requires the chain rule, which rule will certainly be appropriate for the second derivative?

6. What is the relation between factorials and higher derivatives?

7. What quadratic expression represents the value of $\dfrac{(n+1)!}{(n-1)!}$?

Proof questions:

1. Assume that $f(x)$ is a function with both first and second derivative. What is the second derivative of the function $y = x\, f(x^2)$? Collect like terms and factors in the final answer.

2. When computing the general formula for $f^{(n)}(x)$, you may run into a product of the form $3\times5\times7\times\cdots\times(2n-1)$, that is, the product of the first *n* odd numbers. For our purposes, it is enough to denote such product as I have just done, but you may wonder if there is a factorial-based way to represent this number. Of course it is NOT $(2n-1)!$, since such formula includes all even numbers, and some authors denote it by $(2n-1)!!$ (*double factorial*). But there is a nice formula for it:

$$1\times3\times5\times7\times\cdots\times(2n-1) = \frac{(2n-1)!}{2^{n-1}(n-1)!}$$

For some weekend fun, you task is to show that this formula is correct by using the method of induction, that is, by showing that:

a) The formula works for n=1.
b) If the formula works for any n, it also works for $n+1$.

Application questions:

1. What is the acceleration of an object moving on the x axis so that its position at time t is $x = \sqrt[3]{2t-5}$?

2. An object falls through a force field so that its height is given by the function $h(t) = \dfrac{20}{(2+t)^2}$ in meters and seconds. Compute the formula that provides the acceleration of this object in terms of time.

What questions do you have for your instructor?