Chapter 5: Derivatives of transcendental functions — **Section 1**

Derivatives of Logarithmic functions

What you need to know already:	What you can learn here:
➤ Definition of derivative and all basic differentiation rules.	➤ How to differentiation a logarithmic function.

In the section on inverse functions I included, as an example, the formula for the derivative of the natural logarithm. Here it is for your convenience:

Technical fact:

The natural logarithm rule

If $y = \ln x$, then $y' = \dfrac{1}{x}$

Proof

By using the method of implicit differentiation, we start from our function and apply its inverse, the exponential, to both sides:

$$y = \ln x \quad \Rightarrow \quad e^y = x$$

Next we differentiate implicitly both sides and isolate y':

$$\Rightarrow \quad e^y y' = 1 \quad \Rightarrow \quad y' = \frac{1}{e^y}$$

But we have just seen that $e^y = x$, so that we can conclude that:

$$(\ln x)' = \frac{1}{x}$$

as claimed.

Example: $y = \ln\left(x + \ln\left(x^2+1\right)\right)$

The presence of the natural logarithm suggests the use of the corresponding rule, but we need to also use the chain rule, since this is a composite function:

$$y' = \frac{1}{x + \ln\left(x^2+1\right)}\left(x + \ln\left(x^2+1\right)\right)'$$

We can now use appropriate rules to differentiate inside the bracket, including using the one for the natural logarithm again, thus concluding that:

$$y' = \frac{1}{x + \ln\left(x^2+1\right)}\left(1 + \frac{2x}{x^2+1}\right)$$

The formula for the derivative of the *natural* logarithm can be easily extended to a formula for the derivative of *any* logarithmic function.

Technical fact:

The general logarithm rule

If $y = \log_a x$, with $a > 0$, then $y' = \dfrac{1}{x \ln a}$

Proof

By using the change of base formula, we see that:

$$y = \log_a x = \frac{\ln x}{\ln a}$$

But $\ln a$ is a constant, so we can combine the logarithm rule and the coefficient rule to obtain:

$$y' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x \ln a}$$

Example: $f(x) = \dfrac{\log(x^2+3)}{\log_2(x+1)}$

Since this function consists of a quotient, we start with the quotient rule. Within it we then use the general logarithm rule and other suitable ones:

$$f'(x) = \frac{\left[\log(x^2+3)\right]' \log_2(x+1) - \left[\log_2(x+1)\right]' \log(x^2+3)}{\left[\log_2(x+1)\right]^2}$$

$$f'(x) = \frac{\dfrac{2x\log_2(x+1)}{(x^2+3)\ln 10} - \dfrac{\log(x^2+3)}{(x+1)\ln 2}}{\left[\log_2(x+1)\right]^2}$$

If you prefer, we can avoid using different formulae for different logarithm and start by changing all to natural logarithms:

$$f(x) = \frac{\log(x^2+3)}{\log_2(x+1)} = \frac{\dfrac{\ln(x^2+3)}{\ln 10}}{\dfrac{\ln(x+1)}{\ln 2}} = \frac{\ln 2}{\ln 10}\frac{\ln(x^2+3)}{\ln(x+1)}$$

The first fraction is now a constant coefficient that does not affect the derivative, so we focus on the remaining fraction with the natural logarithm rule and others as needed:

$$f'(x) = \frac{\ln 2}{\ln 10}\frac{\left[\ln(x^2+3)\right]' \ln(x+1) - \left[\ln(x+1)\right]' \ln(x^2+3)}{\left[\ln(x+1)\right]^2}$$

$$= \frac{\ln 2}{\ln 10}\frac{\dfrac{2x\ln(x+1)}{x^2+3} - \dfrac{\ln(x^2+3)}{x+1}}{\left[\ln(x+1)\right]^2}$$

Which of the two approaches shown in this example is better?

That is up to you, depending on which method makes more sense and is clearer to you. As in all other similar situations, any valid method is acceptable; just don't use an invalid one!

I conclude this short section with the proof of an interesting fact that you may have seen previously.

Technical facts

- $\lim_{x \to 0}(1+x)^{\frac{1}{x}} = e$
- $\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{x} = e$

Proof

Now we know that the derivative of $f(x) = \ln x$ is $f'(x) = \frac{1}{x}$.

Therefore:

$$1 = f'(1) = \lim_{h \to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h \to 0}\frac{\ln(1+h) - \ln 1}{h}$$

By using basic properties of the natural logarithm, we can see that:

$$1 = \lim_{h \to 0}\frac{\ln(1+h)}{h} = \lim_{h \to 0}\ln(1+h)^{\frac{1}{h}}$$

If we take exponentials on both sides and use the continuity of the exponential function, we obtain:

$$e = e^{\lim_{h \to 0}\ln(1+h)^{\frac{1}{h}}} = \lim_{h \to 0} e^{\ln(1+h)^{\frac{1}{h}}} = \lim_{h \to 0}(1+h)^{\frac{1}{h}}$$

If, in this formula, we use $x = h$ we obtain the first limit:

$$e = \lim_{x \to 0}(1+x)^{\frac{1}{x}}$$

On the other hand, if we let $x = \frac{1}{h}$ we obtain the second:

$$e = \lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{x}$$

By the way, the latter limit reflects the way in which, historically, the number *e* was discovered. It came from the world of finances and it is an interesting story, but beyond our current goals.

Summary

- The derivative of any logarithmic function can be obtained by considering it as the inverse of the corresponding exponential, or by using the change of base formula.

Common errors to avoid

- Remember that the simple rule only applies to the natural logarithm: do not apply it in the same way to other logarithms.
- The logarithm is a single function, so when it is combined with other functions other appropriate rules may also apply, such as product, chain etc.

Learning questions for Section D 5-1

Review questions:

1. Describe how to differentiate a function that involves a logarithm.

Memory questions:

1. What is the derivative of the function $y = \ln x$?

2. What is the derivative of the function $y = \log_a x$?

3. What is the value of $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$?

Computation questions:

Compute the derivatives of the functions presented in questions 1-12.

1. $y = \ln\left(-x^3\right)$

2. $y = \ln\left(1 - x^2\right)$

3. $y = \ln\left(3x + e^{x^2+1}\right)$

4. $y = \ln\left(\frac{3x}{e^{x^2+1}}\right)$

5. $y = \log_{x^2}\left(3 + x^2\right)$

6. $y = \log_{x^2}\left(e^x + x^2\right)$

7. $y = \log_{10} \frac{10x}{5x+3}$

8. $y = \log_{\pi}\left(\frac{x^2+1}{x+1}\right)$

9. $y = \frac{\ln x}{\ln(x+1)}$

10. $y = \ln \frac{x^2+3}{e^{\sqrt{x}-5}}$

11. $y = xe^{x^2} + \frac{x}{\ln x}$

12. $y = \frac{x}{e^{x^2}} + \frac{\ln x}{x}$

13. Compute the second derivative of the function $y=\ln\left(x-\sqrt{x}\right)$.

14. Compute the second derivative of the function $y=\ln\left(x^2-\frac{1}{\sqrt{x}}\right)$.

15. Compute $\frac{d^2y}{dx^2}$ if y is defined by the equation $\ln\left(x^2+3\right)+3\ln y=5$.
Notice that there are two ways to answer this question: both are acceptable.

Theory questions:

1. What method is use to obtain the derivative of the natural logarithm?

2. What is the derivative of $\ln kx$, for any positive number k?

Proof questions:

1. Show that $\frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x}$ by using the fact that $|x|=\sqrt{x^2}$.

2. Show that the general logarithm rule applies also to the natural logarithm, in which case it reduces to the natural logarithm rule.

Application questions:

1. What is the equation of the line tangent to the curve $y^4=\left(\ln ex\right)\left(2^x+\sqrt[8]{x^3}\right)$ at the point $\left(1,-\sqrt[4]{3}\right)$?

Templated questions:

1. Construct a function that involves a logarithm and compute its derivative

What questions do you have for your instructor?