Logarithmic differentiation

What you need to know already:	*What you can learn here:*
➤ All basic differentiation rules, implicit differentiation and the derivative of the natural logarithm.	➤ How to compute derivative of certain complicated functions for which the logarithm can provide a simpler method of solution.

There are many functions for which the rules and methods of differentiation we have seen so far are not sufficient. But for some of them the properties of logarithms can be of assistance. For instance, how would you compute the derivative of the function $y = x^x$?

Power rule? Exponential rule?

Think about it: neither!

Warning bells

A function of the form $y = \left(f(x)\right)^{g(x)}$:

- ***cannot*** be differentiated by using the ***power*** rule, because its exponent is not constant, and
- ***cannot*** be differentiated by using the ***exponential*** rule, because its base is not constant.

We seem to be in a bind! To develop a method of differentiation that works for this case, we can use the fact that whenever $a > 0$, $\ln a^b = b \ln a$, that is, the fact that logarithms allow us to change exponents to the role of factors.

The strategy of logarithmic differentiation

To differentiate a function of the form $y = \left(f(x)\right)^{g(x)}$

1. ***Apply logarithms*** to both sides of the equation, so as to change it to $\ln y = g(x) \ln f(x)$
2. ***Apply implicit differentiation*** on the left and the ***product rule*** on the right of this equation
3. ***Solve*** for *y*'.
4. If needed, ***replace*** *y* in the solution with the original function.

Example: $y = x^x$

We apply logarithms on both sides:

$$\ln y = \ln x^x \quad \Rightarrow \quad \ln y = x \ln x$$

Now we use implicit differentiation and the product rule on the right side

$$\frac{1}{y} y' = (x)' \ln x + (\ln x)' x = \ln x + 1$$

Now we solve for *y*:

$$\frac{y'}{y} = \ln x + 1 \quad \Rightarrow \quad y' = y(\ln x + 1)$$

And finally replace *y* with its original formula

$$\Rightarrow \quad y' = x^x (\ln x + 1)$$

For a function of the form $y = (f(x))^{g(x)}$ logarithmic differentiation is the only way to compute the derivative. However, if we implement it in the most general situation, it is possible to obtain a formula that can be used directly.

Technical fact

The Powex rule

$$\frac{d}{dx}(f(x))^{g(x)} = g' \, f^g \ln f + g \, f^{g-1} f'$$

Proof

To simplify the notation, I will drop *x* in the proof , as I have done in the formula, but remember that *f* and *g* both represent functions of *x*.

So, if we start from $y = f^g$, we can then:

- Take logarithms on both sides: $\ln y = \ln f^g = g \ln f$
- Differentiate both sides: $\frac{y'}{y} = g' \ln f + \frac{g \, f'}{f}$
- Solve for *y*' and replace *y*: $y' = f^g \left(g' \ln f + \frac{gf'}{f} \right)$

If we expand the right side, we arrive at the conclusion:

$$y' = g' \, f^g \ln f + g \, f^{g-1} f'$$

So? How does that help?

Well, I said that a formula can be constructed; I did not say it was a simple formula. But this rule becomes clearer if you consider this way to interpret it.

Knot on your finger

The powex rule is telling us that to differentiate a function of the form $y = (f(x))^{g(x)}$, we can:

- Differentiate it by ***pretending*** that the ***base is constant***, thus using chain and exponential rules
- Differentiate it by ***pretending*** that the ***exponent is constant***, thus using chain and power rules
- ***Add*** the two terms obtained in this way.

Example: $y=\left(x^2+1\right)^{2x^3}$

We differentiate this function pretending that the base is constant, thus using chain and exponential rules:

$$y_1=\left[\left(x^2+1\right)^{2x^3}\ln\left(x^2+1\right)\right]\left(6x^2\right)$$

Next we differentiate it by pretending that the exponent is constant, thus using chain and power rules:

$$y_2=\left[2x^3\left(x^2+1\right)^{2x^3-1}\right]\times 2x$$

The derivative we are looking for is the sum of these two pieces:

$$y'=6x^2\left(x^2+1\right)^{2x^3}\ln\left(x^2+1\right)+4x^4\left(x^2+1\right)^{2x^3-1}$$

Notice that for functions of the form $y=\left(f(x)\right)^{g(x)}$, the method of logarithmic differentiation, or the powex rule it produces, is the only way to go. However, a different property of logarithms provides an alternative way to compute the derivative of functions that consist of complicated products or quotients.

Alternative strategy for differentiating large products: Logarithmic differentiation

To compute the derivative of a function of the form $y=\dfrac{f(x)g(x)}{b(x)}$, we can use quotient and product rules, or we can:

1. ***Apply logarithms*** to both sides of the equation and change it to:
   $$\ln y=\ln f(x)+\ln g(x)-\ln b(x)$$
2. If possible, ***use properties of logarithms*** to simplify each term further.
3. Apply ***implicit differentiation*** to the equation so obtained and thus solve for y'.

Example: $y=\dfrac{\sqrt[3]{x^2-1}\,e^{x^3}}{x^3+x}$

It is possible to differentiate this function by using quotient rule, followed by product rule for the top and then all other required rules. But we can also use logarithmic differentiation instead.

We start by applying logarithms to both sides:

$$\ln y=\ln\left(\frac{\sqrt[3]{x^2-1}\,e^{x^3}}{x^3+x}\right)$$

We can now use properties of the logarithm to change the expression on the right to one that is easier to differentiate:

$$\ln y=\ln\sqrt[3]{x^2-1}+\ln e^{x^3}-\ln\left(x^3+x\right)$$

$$\Rightarrow\quad \ln y=\frac{1}{3}\ln\left(x^2-1\right)+x^3-\ln\left(x^3+x\right)$$

Next we use implicit differentiation and solve for y':

$$\Rightarrow \quad \frac{y'}{y} = \frac{2x}{3\left(x^2-1\right)} + 3x^2 - \frac{3x^2+1}{x^3+x}$$

$$\Rightarrow \quad y' = y\left(\frac{2x}{3\left(x^2-1\right)} + 3x^2 - \frac{3x^2+1}{x^3+x}\right)$$

$$= \frac{\sqrt[3]{x^2-1}\, e^{x^3}}{x^3+x}\left(\frac{2x}{3\left(x^2-1\right)} + 3x^2 - \frac{3x^2+1}{x^3+x}\right)$$

And is this method simpler?

That depends on you and on the function you are dealing with. I leave to you, your experience (make sure to acquire a good one!) and your preferences the decision of whether to use basic rules or logarithmic differentiation for functions of this form.

Summary

- The method of logarithmic differentiation is needed to compute the derivative of functions that have the variable in both the base and the exponent of the same power.
- The method of logarithmic differentiation gives rise to the powex rule, used to compute the derivative of functions of the form $y = \left(f(x)\right)^{g(x)}$.
- The method of logarithmic differentiation may also be used to compute the derivative of functions that consist of large and/or complicated products and fractions.

Common errors to avoid

- Do not use either the power rule alone or the exponential rule alone for functions of the form $y = \left(f(x)\right)^{g(x)}$: they need logarithmic differentiation or the powex rule.

Learning questions for Section D 5-2

Review questions:

1. Describe when logarithmic differentiation is needed and when it is just useful.
2. Describe how logarithmic differentiation works.

Memory questions:

1. For what types of function is logarithmic differentiation required?
2. Which rule provides the derivative of a function of the form $y = f(x)^{g(x)}$?
3. For what types of function is logarithmic differentiation useful, but not necessary?

Computation questions:

For each of the functions presented in questions 1-23, decide whether the method of logarithmic differentiation is needed or simply useful to compute the derivative, then apply it and obtain the derivative in question.

1. $y = x^{\ln x}$
2. $y = (\ln x)^x$
3. $y = \ln x^x$
4. $y = \ln x^{\ln x}$
5. $y = (\ln x)^{\ln x}$
6. $y = (\ln x)^{\frac{1}{x}}$
7. $f(x) = \left(x^3 - 2x\right)^{\left(x - 2x^3\right)}$
8. $y = \left(e^x + \ln x\right)^x$
9. $y = e^x x^e 3^{x+e}$
10. $y = e^{3x}\left(x^3 + x\right)^2 \left(x + \ln x\right)^5$
11. $y = \left(x - \sqrt{x}\right)^{x - \sqrt{x}}$
12. $y = \left(x^2 - 2x\right)^{\ln(x-3)}$
13. $y = (x+1)^{x+3}$

14. $y = \ln^x\left(x^2+1\right)$

15. $y = \dfrac{(x^2+1)\ln x}{e^{3x-6}}$

16. $y = \dfrac{\sqrt[3]{x}e^{x^3+x}}{\ln\left(e^x+x\right)}$

17. $y = \dfrac{e^{2x}}{x^2\ln x}$

18. $y = \dfrac{e^{3x}}{\left(x^2+1\right)\ln x}$

19. $y = \sqrt[3]{x}\, e^{x^3}(x^3+x)$

20. $y = \ln\left(x + x^{\sqrt{x}}\right)$

21. $y = \dfrac{e^x \ln x}{x^{100}}$

22. $y = \dfrac{x^{2x}}{(\ln x)^x}$

Theory questions:

1. Is logarithmic differentiation a rule or a method?
2. Which method of differentiation begins by applying the logarithmic function?
3. What is the first step to take when computing the derivative of a function of the form $y = \log_{f(x)} g(x)$?
4. What happens to the slope of the natural logarithmic function as x increases?

Proof questions:

1. Use the method of logarithmic differentiation to prove the power rule in its generality.
2. Show that the *Powex rule* can be obtained by writing a function of the form $y = \left(f(x)\right)^{g(x)}$ as $y = e^{g(x)\ln f(x)}$.
3. Have you ever been asked to determine, without a calculator, which of 6^7 or 7^6 (or similar numbers) is bigger? You can now prove a little trick to answer the question, at least in the form that is usually asked. To get there, follow these steps:

   a) Prove that the function $f\left(x\right) = x^{\frac{1}{x}}$, which is defined only for $x > 0$, is increasing if $x < e$ and decreasing after that.

b) Prove that for two positive numbers a and b, the statements $a^b > b^a$ is true if an only if $a^{\frac{1}{a}} > b^{\frac{1}{b}}$.

c) Conclude that if $e \le a < b$ then $a^b > b^a$

Therefore, as long as the two numbers interchanged are both greater than e, the one with the smaller base will give the bigger power: $a^b > b^a$. In particular, $6^7 > 7^6$.

Application questions:

1. Determine the equation of the line tangent to the curve $y = e^x\left(x^2+1\right)^{\ln e^2 x}$ at $x=1$.

2. Determine the equation of the line tangent to the function $f\left(x\right)=\left[\left(x+1\right)^4+1\right]^{\ln\left(x+1\right)}$ at its *y*-intercept.

Templated questions:

1. Construct a function of the form $y = f\left(x\right)^{g(x)}$ and then compute its derivative by using both logarithmic differentiation and the Powex rule. Check that the conclusions are equivalent.

2. Construct a function consisting of a large (but not too much!) product and quotient and compute its derivative by using both basic rules and implicit differentiation. Check that the answers agree.

What questions do you have for your instructor?