Chapter 5: Derivatives of transcendental functions — *Section 4*

Derivatives of

Inverse hyperbolic functions

What you need to know already:	*What you can learn here:*
➤ All basic differentiation rules, the derivatives of hyperbolic functions and the method of implicit differentiation.	➤ How to differentiate inverse hyperbolic functions.

As you may remember, inverse hyperbolic functions, being the inverses of functions defined by formulae, have themselves formulae. Here they are, for your convenience.

Technical fact

The formulae of the basic inverse hyperbolic functions are:

$$\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Will we use these formulae to obtain their derivatives?

You will, as part of your *Learning questions*. But one can also use the method of implicit differentiation, since they are inverse functions. Either way, we obtain the following formulae:

Technical fact

The derivatives of the basic inverse hyperbolic functions are:

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}, \quad -\infty < x < \infty$$

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}, \quad -1 < x < 1$$

Proof

I will show you how to prove the formula for the inverse hyperbolic sine. The others are found in the same way and I leave that to you.

We start from $y = \sinh^{-1} x$ and apply the hyperbolic sine function to both sides of the equation:

$$y = \sinh^{-1} x \quad \Rightarrow \quad \sinh y = x$$

Now we differentiate both sides, implicitly:

$$\Rightarrow \quad y' \cosh y = 1$$

We now solve for *y*' and remember that the hyperbolic cosine is always positive:

$$\Rightarrow \quad y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\cosh^2 y}}$$

Finally, we use the basic hyperbolic identity and the fact that $\sinh y = x$:

$$\Rightarrow \quad y' = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

This is the stated formula.

These formulae also look similar to those of the corresponding trigonometric functions!

Yes, so once again you may want to focus on the differences, as well as the similarities.

Since I leave to you the pleasure to prove the other two formulae, I will finish this section with an example of how to use these formulae within a larger function.

Example: $f(x) = 5x^2 \cosh^{-1}(x^3)$

This is a product, so we start with the product rule:

$$\frac{dy}{dx} = \left(5x^2\right)' \cosh^{-1}\left(x^3\right) + 5x^2 \left(\cosh^{-1}\left(x^3\right)\right)'$$

Now we use coefficient and power rule for the first bracket and chain plus hyperbolic rule for the second:

$$\frac{dy}{dx} = 10x \cosh^{-1}\left(x^3\right) + 5x^2 \left(\frac{1}{\sqrt{\left(x^3\right)^2 - 1}} 3x^2 \right)$$

If desired, we can combine the powers in the second term:

$$\frac{dy}{dx} = 10x \cosh^{-1}\left(x^3\right) + \frac{15x^4}{\sqrt{x^6 - 1}}$$

Summary

- The formulae for the derivatives of inverse hyperbolic functions may be obtained either by using their defining formulae, or by using the method of implicit differentiation.

Common errors to avoid

- Don't rely on your memory all the time to remember the formulae of this section. They are not used often enough for your brain to keep them in permanent storage, therefore, review them each time you need them, and make sure to review them well before a test.

Learning questions for Section D 5-4

Review questions:

1. Explain how to obtain the formulae for the derivatives of the inverse hyperbolic functions.

Memory questions:

1. What is the derivative of the function $y = \sinh^{-1} x$?

2. What is the derivative of the function $y = \cosh^{-1} x$?

3. What is the derivative of the function $y = \tanh^{-1} x$?

Computation questions:

Compute the derivative of the basic inverse hyperbolic functions presented in questions 1-3 by using both implicit differentiation and the logarithmic formula that defines them. Check that the two answers are consistent.

1. $y = \sinh^{-1}(x)$

2. $y = \cosh^{-1}(x)$

3. $y = \tanh^{-1}(x)$

For each of the functions presented in questions 4-12:

a) Compute its derivative and, unless this is horribly complicated, compute the second derivative
b) Construct the equation of the tangent line at one intercept, or, if neither exist, at a point of your choice
c) Compute the formula for the differential *dy*.

4. $y = \cosh\left(\sinh^{-1} x\right)$

5. $y = \sinh^{-1}\left(\cos^2 x\right)$

6. $y = \ln \sinh x - \cosh^{-1}\left(e^{x^2}\right)$

7. $f(x) = \sin\left(\cosh x \tanh^{-1}\left(\dfrac{x^2}{2}\right)\right)$.

8. $y = \sinh^{-1}\left(x^3 + e^x\right)$

9. $f(x) = x^2 \cosh^{-2}\left(1 + \ln^2 x\right)$

10. $y = \dfrac{\sinh x}{\cosh^{-1} x}$

11. $f(x) = \dfrac{2^{-x} \sinh^{-1}\left(1 + \ln x\right)}{\sqrt{1+x}}$

12. $f(x) = \left(\tanh^{-1} x\right)^{\tan x}$

13. $y = \tanh^{-1}\left(e^{3x} - \sqrt{x+4}\right)$

14. $y = \tan^{-1}\left(\ln^3 x - \sqrt[5]{x+4}\right)$

Theory questions:

1. Which nice feature is common to the derivatives of both inverse hyperbolic and inverse trigonometric functions?

2. How are the derivatives of the inverse hyperbolic tangent and inverse tangent different?

3. What methods can be used to compute the derivatives of inverse hyperbolic functions?

Templated questions:

1. Construct a simple function involving inverse hyperbolic functions and:
   a) Compute its derivative
   b) Unless the first derivative is horribly complicated, compute the second derivative
   c) Construct the equation of the tangent line at one intercept, or, if neither exist, at a point f your choice
   d) Compute the formula for the differential *dy*.

What questions do you have for your instructor?