Chapter 5: Derivatives of transcendental functions | **Section 5**

Derivatives of Trigonometric functions

What you need to know already:	*What you can learn here:*
➤ Basic trigonometric limits, the definition of derivative, all basic differentiation rule.	➤ How to differentiate functions involving trigonometric functions.

None of the differentiation rules we have developed so far can be used to compute the derivatives of trigonometric functions. To do that, we need to go back to the definition of derivative.

Technical fact

If $y = \sin x$, then $y' = \cos x$

If $y = \cos x$, then $y' = -\sin x$

Proof

I will show you why the first derivative formula is correct and leave the other to you in the *Learning questions*.

By definition:

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

By using the addition formula for the sine function, this becomes:

$$= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$

Now we split the fraction in two pieces by using the common factor of $\sin x$:

$$= \lim_{h\to 0}\frac{\sin x\cos h - \sin x}{h} + \lim_{h\to 0}\frac{\sin h\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\sin h\cos x}{h}$$

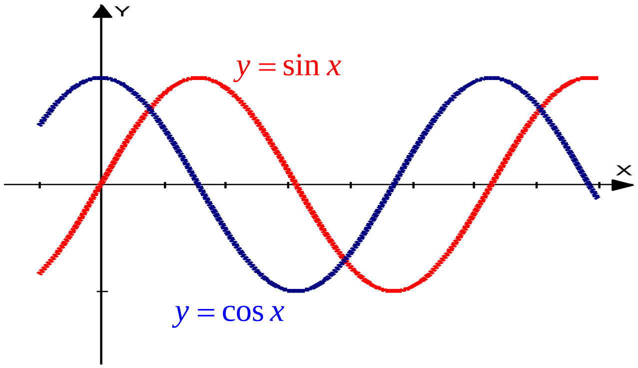
Since the limit is computed as *h* approaches 0, the factors $\sin x$ and $\cos x$, which do not involve *h*, can be moved out of the limit:

$$= \sin x\lim_{h\to 0}\frac{(\cos h - 1)}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}$$

But now we are left with the two fundamental trigonometric limits, whose values we know! Therefore, we can reach the conclusion:

$$\frac{d}{dx}\sin x = \sin x \times 0 + \cos x \times 1 = \cos x$$

Incidentally, the fact that $(\sin x)' = \cos x$ can be intuited by comparing their graphs:



We can see that the sine function starts with a slope of 1 that gradually decreases to 0 and then becomes negative, only to switch again and become positive. This is definitely NOT a proof, but it is an observation that was made and used, well before calculus was invented, by Islamic astronomers in their calculations of celestial motions. Of course, the same astronomical motivation led Newton to modern calculus: some things connect repeatedly!

Now that the two basic formulae are in place we can use them, together with other suitable differentiation rules, to obtain the derivatives of the other four basic trigonometric functions.

Technical fact

If $y = \tan x$, then $y' = \sec^2 x$

If $y = \sec x$, then $y' = \sec x \tan x$

If $y = \csc x$, then $y' = -\csc x \cot x$

If $y = \cot x$, then $y' = -\csc^2 x$

Proof

As before, I will show you how to prove the first formula and leave the others to you. By definition:

$$\tan x = \frac{\sin x}{\cos x}$$

We can therefore use the quotient rule and the trig derivatives we have just discovered:

$$(\tan x)' = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

By using the basic Pythagorean identity we can conclude that:

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = \sec^2 x$$

And now that we have our six basic formulae, we can combine them with all other rules and methods we know already to differentiate any function that involves trigonometric parts. I will show you one example and give you some more in the *Learning questions* for your pleasure.

Example: $y = \tan\left(\sin 3x + x^2\right)$

Let's find the second derivative of this function!

For the first derivative, we start by using the chain rule and the derivative of the tangent function:

$$y' = \sec^2\left(\sin 3x + x^2\right) \times \left(\sin 3x + x^2\right)'$$

Next we use suitable rules, including that for the derivative of the sine function, to complete the computation:

$$y' = \sec^2\left(\sin 3x + x^2\right) \times \left(3\cos 3x + 2x\right)$$

For the second derivative we start with the product rule and then apply all suitable rules:

$$y'' = \left[\sec^2\left(\sin 3x + x^2\right)\right]'\left(3\cos 3x + 2x\right) +$$

$$+\sec^2\left(\sin 3x + x^2\right)\left(3\cos 3x + 2x\right)'$$

$$= \left[2\sec\left(\sin 3x + x^2\right)\times\left[\sec\left(\sin 3x + x^2\right)\right]'\right]\left(3\cos 3x + 2x\right) +$$

$$+\sec^2\left(\sin 3x + x^2\right)\left(-9\sin 3x + 2\right)$$

We need two more applications of the chain rule to complete the work:

$$= 2\left(3\cos 3x + 2x\right)^2\sec^2\left(\sin 3x + x^2\right)\tan\left(\sin 3x + x^2\right) +$$

$$+\sec^2\left(\sin 3x + x^2\right)\left(2 - 9\sin 3x\right)$$

That looks long and ugly, but, thank goodness, we are done!

Summary

- The derivatives of $y = \sin x$ and $y = \cos x$ are obtained by using the definition of derivative.
- The derivatives of $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \csc x$ are obtained by using their defining identities and basic differentiation rules.

Common errors to avoid

- It is easy to forget the derivatives of trigonometric functions, especially the four defined by fractions. Practice enough on them so as to memorize them effectively.

Learning questions for Section D 5-5

Review questions:

1. Explain how the rules to differentiate basic trigonometric functions are obtained.

Memory questions:

1. What is the derivative of $f(x) = \sin x$?

2. What is the derivative of $f(x) = \cos x$?

3. What is the derivative of $f(x) = \tan x$?

4. What is the derivative of $f(x) = \cot x$?

5. What is the derivative of $f(x) = \sec x$?

Computation questions:

Use appropriate rules of differentiation to obtain the derivatives of the functions presented in questions 1-50:

1. $y = \sin^2 x$
2. $y = \cos x^3$
3. $y = \cos^3 x$
4. $y = x^4 \cos x$
5. $y = x^2 \tan x^2$
6. $y = x^{10} \sec x$
7. $y = (3x^3 - 5\sin x)^4$
8. $y = 4x^2 \sin^4 x + 4\tan 2x$
9. $y = (x+1)^3 \cos(3x-5)$
10. $y = \dfrac{\tan x}{e^x}$
11. $y = \sqrt[6]{\cos(x^3 + x)}$
12. $y = \dfrac{2 - x\sqrt{e^x}}{\cos\sqrt{x}}$
13. $y = e^{-x}\sin(e^x)$
14. $y = e^{\cos x} - 2\cos e^x$
15. $y = \dfrac{e^{\cos x}}{\cos e^x}$
16. $y = \dfrac{\sin e^x}{x^3 \cos 2x}$
17. $y = \tan\dfrac{2x+1}{\cos x}$
18. $y = \dfrac{\tan\sqrt{x}}{\sin 2^x}$
19. $y = \dfrac{e^x + 3x^3}{\cos x}$
20. $y = \dfrac{\sin e^x}{x^2 \cos 2x}$
21. $y = \dfrac{e^{\sin x}}{x^2 \cos x^2}$
22. $y = \dfrac{\sqrt{\tan x^2}}{\sin \log x}$
23. $y = \dfrac{\sin \pi x}{\cosh ex - \ln^2 x^2}$
24. $y = \dfrac{\sinh^2 x}{\cos \pi x - \ln^3 2x^2}$
25. $y = \sqrt{\tan x^2 + \ln\left(x + e^x\right)}$
26. $y = \cosh\sqrt{x^2+1} - \dfrac{\sin e^x - e^{\cos x}}{\ln x^2}$

27. $y=\dfrac{\tan\dfrac{2}{x}}{e^{\sin x}}$

28. $y=\dfrac{e^{x-6}\sin x}{x^2+x}$

29. $y=(x+1)^{\cos x}$

30. $y=\ln x^{\cos x}$

31. $y=(\sin x)^{\ln x}$

32. $y=\cos\left(x^{\ln x}\right)$

33. $y=\ln\left(x+x^{\cos x}\right)$

34. $y=(x+3)^{\cos x}$

35. $y=\sin^2 x^{e^x}$.

36. $y=\left(x^2\right)^{\sin x^2}$

37. $y=\left(\sin x^2\right)^{x^2}$

38. $y=\left(x^2\right)^{\cos x^2}$

39. $y=\ln^{\cos x^2} x$

40. $y=\ln^{\sin x^2} x$

41. $y=\sqrt[x]{\cos x}$

42. $y=\dfrac{e^{3x}}{x^2+1}+\sqrt{\sec x+x}\,(\cosh x)$

43. $y=\dfrac{(x^2+1)\cos x}{e^{3x-6}}$

44. $y=\dfrac{e^{2x}}{\cos x\ln x}$

45. $y=\dfrac{\ln \pi x+\cosh^2 x-\sinh^2 x}{\sin \pi^x+\cos x^e}$

46. $y=\dfrac{\ln x+\cos^2 x-\sin^2 x}{\sinh \pi^x+\cosh x^e}$

47. $y=\cos\sqrt{1-x^2}-\dfrac{\sinh e^x-e^{\cos x}}{\ln\left(x^2+1\right)}$

48. $y=\sqrt{\sin\left(1-x^2\right)}-\dfrac{\sin e^x-e^{\cosh x}}{\ln\left(x^2+1\right)}$

49. $y=e^{\cos x}+\cos\left(e^x\right)$

50. $y=\cos\left(x^e\right)-x^{\cos e}$

Compute the second derivative of the functions in questions 51-56:

51. $y=\ln\left(\sin^2 x-\cos^2 x\right)$

52. $f(x)=x\sin x^2$

53. $y=\ln\sin x$

54. $y=\ln\left(\cos(x)\right)$

55. $y=\sin\left(\ln(x)\right)$

56. $y=\cos\ln x$

57. Determine $f'''\left(\frac{\pi}{8}\right)$ for $f(x)=\tan(2x)$

58. Use implicit differentiation to determine the second derivative of the function defined by the equation $\sin(x)+\sin(y)=1$.

59. Determine the 24th derivative of $y=\frac{\sin x}{e^x}$

60. Compute the differential of $y=\frac{\cos x}{1+x}$ at $x=0$ with $dx=\frac{\pi}{36}$

61. Compute the derivative of the function $y=\cos\left(x^x\right)$

62. If $f(x)$ is a differentiable function, what is the second derivative of $g(x)=f\left(x^2\cos x\right)$?

63. Compute $\frac{d^2y}{dx^2}$ for the curve defined by $x\ln y - y\cos(x)=y^2$. Express your answer in terms of *x y* and *y'*, but no need to beautify the expression.

Theory questions:

1. Which basic limit formulae are needed in the proof of the derivative formula for the sine function?

2. Which differentiation rule is used to obtain the formula for the derivative of $y=\tan x$?

3. What is the 100th derivative of $y=\sin x$?

4. Which trig identity is used to prove the derivative formula for the sine function?

5. The limit $\lim_{h\to 0}\frac{\sinh}{h}$ that is needed to compute derivatives of trig functions is itself a derivative. Of what function and at what *x* value?

6. What is the derivative of $f(x)=\ln\left(\ln(\sin x)\right)$?

Proof questions:

1. Use the definition of derivative to obtain the derivative of $y=\cos x$

2. Use the definition of derivative to prove that $\left(\sin 2x\right)'=2\cos 2x$

3. Use appropriate rules of differentiation to obtain the derivative of $y = \sec x$ and $y = \csc x$

4. Prove that $\frac{d}{dx}(\sec x) = \sec x \tan x$ without using the quotient rule.

5. Show that line tangent to the curve $y = \frac{\cos^2 x}{2+\sin x^2}$ at its *y*-intercept is horizontal.

6. Compute the derivative of the function $y = \frac{\sin x}{x}$ by using the definition of derivative and check that this is correct by using appropriate differentiation rules.

Application questions:

Determine the equation of the line tangent to each of the curves presented in questions 1-12 at the given point.

1. $\tan(x+y) = x$ at (0, 0)

2. $y = \frac{\sin x}{x}$ at $(\pi, 0)$?

3. $y = \tan^2 x \sin 2x$ at $x = \frac{\pi}{4}$.

4. $y = e^x - \sin x$ at (π, e^π)

5. $y^2 = \cos 2x\left(e^x + \sqrt{x+1}\right)$ at $\left(0, -\sqrt{2}\right)$?

6. $y = \frac{e^{\sin x}\ln\left(x^2+e^x\right)}{x^4+1}$ at the y intercept .

7. $\sin x^2 + y^3 = \ln\frac{ex}{\sqrt{\pi}}$ at $\left(\sqrt{\pi}, 1\right)$

8. $y = \tan^2\left(\frac{\pi}{2}\sin\frac{x}{2}\right)$ at $x = \frac{\pi}{3}$.

9. $y = \frac{e^{\sqrt{\cos x}+x} + \tan(x^2)}{x^{10}+\cos x}$ at *x*=0.

10. $y^2 = x^2 + \sin xy$ at $\left(\sqrt{\pi}, \sqrt{\pi}\right)$

11. $f(x) = \sec x^2 - \tan^2 3x$ at $(0, 1)$.

12. $f(x) = \frac{x^2\cos x^2}{e^{\sin x}}$ at the y-intercept .

13. Assuming, for simplicity, that all months to have 30 days, the amount of daylight in Red Deer is approximately given by the function:

$$L = 12 + 6\sin\left(\frac{2\pi\left(t-80\right)}{365}\right)$$

Compute how much light time we can expect on October 15 and how fast it is changing.

14. An object moves on the x-axis, so that after t seconds it is in position $x = \cos e^t$. What is its acceleration after t seconds?

15. Check whether the lines tangents to $f\left(x\right) = e^{x-2} + x^{-3} - \tan x$ and $g\left(x\right) = \ln x^2 - 5\sinh x$ at $x = 1$ are parallel.

16. An object moves on the x-axis, so that after t seconds it is in position $x = e^{\sin t}$. What is its acceleration after t seconds?

Templated questions:

1. Construct a reasonably simple function involving trigonometric functions and compute its first and second derivatives.

What questions do you have for your instructor?