Chapter 5: Derivatives of transcendental functions — Section 6

Derivatives of
Inverse trigonometric functions

What you need to know already:	What you can learn here:
All basic rules and methods of differentiation, derivatives of trigonometric functions, basic trigonometric identities.	How to compute the derivatives of functions involving inverse trigonometric parts.

Computing the derivatives of inverse trigonometric functions involves the standard use of implicit differentiation that we have applied to other inverse functions, such as the logarithm. All that changes are the identities needed to get a final explicit formula.

As usual, I will show you how it is done in one case and I will let you repeat the process in the other cases.

Technical fact

If $y=\sin^{-1}x$, then $y'=\dfrac{1}{\sqrt{1-x^2}}$.

If $y=\cos^{-1}x$, then $y'=-\dfrac{1}{\sqrt{1-x^2}}$.

If $y=\tan^{-1}x$, then $y'=\dfrac{1}{1+x^2}$.

Proof

This time I will show you how to prove the formula for $y=\cos^{-1}x$.

Before we start, let me remind you that the domain of this function is $[-1,1]$ and its range is $[0,\pi]$.

We begin by applying the cosine function on both sides of the equation:

$$y=\cos^{-1}x \quad\Rightarrow\quad \cos y = x$$

Now we differentiate both sides:

$$\Rightarrow\quad -\sin y\times y'=1 \quad\Rightarrow\quad y'=-\frac{1}{\sin y}$$

We are almost there, except that the right side is still in terms of y and we need it in terms of x. To perform such change, we notice that, since $0\le y\le\pi$ (remember the range above?), we know that $\sin y\ge 0$. Therefore we can write:

$$y'=-\frac{1}{\sin y}=-\frac{1}{\sqrt{\sin^2 y}}$$

We complete the proof by using the basic Pythagorean identity:

$$y' = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$$

The other proofs are done in the same way: just be careful about domains and ranges, so that the roots are used properly!

Armed with these formulae, we can tackle any function that involves inverse trig functions.

Are there derivatives for the inverses of cotangent, secant and cosecant?

Yes, but they are seldom used and involve some subtle technical difficulties, mostly involved with convenient choices of the domain. Therefore, we'll stay away from them. Instead, I will finish with one example that uses the above formulae.

Example: $f(x) = \sqrt{\sin^{-1} 3x + \tan^{-1} x^2}$

In order to keep track of all the parts that make up this function, we may want to insert brackets where they are intended:

$$f(x) = \left(\sin^{-1}(3x) + \tan^{-1}(x^2)\right)^{\frac{1}{2}}$$

Now we are ready to use the chain rule and our new formulae:

$$f'(x) = \frac{1}{2}\left(\sin^{-1}(3x) + \tan^{-1}(x^2)\right)^{-\frac{1}{2}} \times \left(\sin^{-1}(3x) + \tan^{-1}(x^2)\right)'$$

$$f'(x) = \frac{\left(\frac{1}{\sqrt{1-(3x)^2}} \times 3 + \frac{1}{1+(x^2)^2} \times 2x\right)}{2\sqrt{\sin^{-1}(3x) + \tan^{-1}(x^2)}}$$

You may want to clean up this expression as part of your algebra maintenance work ☺.

Notice that when applying the differentiation formula to $\sin^{-1}(3x)$, I squared the whole argument $3x$, since that is the inside of a composition to which I am also applying the chain rule. Don't forget that!

Summary

- Derivatives of inverse trigonometric functions are obtained by using the standard method involving implicit differentiation.

Common errors to avoid

- The formulae for the derivatives of inverse trigonometric functions involve the square of the variable x. This, however, refers to the argument of those functions: when such argument is itself a function of x, the whole expression must be squared when applying the formula.

Learning questions for Section D 5-6

Review questions:

1. Describe the method needed to obtain the derivatives of inverse trigonometric functions.

Memory questions:

1. What is the derivative of the function $y = \sin^{-1} x$?
2. What is the derivative of the function $y = \cos^{-1} x$?
3. What is the derivative of the function $y = \tan^{-1} x$?

Computation questions:

Compute the derivative of each of the functions presented in questions 1-11:

1. $y = x^2 \sin^{-1}(1-x)$.
2. $f(x) = \tan^{-1} x^2 + \dfrac{2}{\tan^{-1} x}$
3. $f(x) = \tan^2 x \left(\tan^{-1} x\right)^2$
4. $f(x) = \tan(3x) \tan^{-1}\left(e^{4x} + x^{4e}\right)$
5. $f(x) = \tan^{-1}(\cos x) + \dfrac{\sin x + \cos x}{\sec x + 1}$
6. $f(x) = \tan^{-1} \sqrt{x^2 + \cos x}$
7. $f(x) = \tan^{-1}\left(\ln x + \sin^{-1} x\right)$
8. $f(x) = \tan^{-1}\left(\dfrac{x^2 \tanh x}{2\sin x + \cos x}\right)$
9. $f(x) = \sin^{-1}\left(\dfrac{x^2 \cosh x}{4\sin x + \ln x}\right)$
10. $f(x) = \tan^{-1}(3x) \tan\left(e^{3x} + x^{3e}\right)$
11. $f(x) = \dfrac{2^{-x} \sin^{-1}(1 + \ln x)}{\sqrt{1+x}}$

12. Determine the first and second derivative of the function $y = x\sin^{-1} x$.

13. Compute the second derivative of the function $y = \sin^{-1}\left(x^2 - \sqrt{x}\right)$.

14. Use both the formulae of this section and implicit differentiation to determine the derivative of the function $y = \sin^{-1}(\tan x)$.

Theory questions:

1. What is the connection between *implicit differentiation* and the topic of this section?

2. Can the power rule and chain rule be used to differentiate $y = \tan^{-1} x$?

3. Are the derivative of the inverse sine, cosine and tangent transcendental, algebraic or rational functions?

Proof questions:

1. Prove the differentiation rule for $y = \tan^{-1} x$.

2. Prove the differentiation rule for $y = \sin^{-1} x$.

Application questions:

1. Determine the rate of change of the function $f(x) = \cos^{-1} x^2 - \tan^{-1} 3x$ at the point $\left(0, \frac{\pi}{2}\right)$.

2. What is the equation of the line tangent to the function $y = \arctan\sqrt{x^2}$ at $x = -0.5$?

3. Determine the slope of the line tangent to the function $y = \sqrt{\sin(2\arcsin\theta)}$ at $\theta = \frac{1}{2}$.

4. What is the slope of the line tangent to the curve $\cosh\left(y^3\right) - 2\tan^{-1}\left(x^2 + y\right) + \frac{\pi}{2} = 1$ at the point of coordinates $(1, 0)$?

What questions do you have for your instructor?