Chapter 6: Derivatives and other types of functions *Section 1*

Partial derivatives

What you need to know already:	***What you can learn here:***
➤ What derivatives are and how to compute them.	➤ How to extend the notion of derivative to functions of several variables.

I hope you still remember the concept of a two-variable function: it is a rule that associates to each pair of input values (x, y) a single output value , usually denoted by $z = f(x, y)$.

We have not seen them in a while, but yes, I remember.

In that case you may remember that limits can be computed for such functions by focussing on one variable at a time. Well, we can do the same with derivatives.

Definition

Given a two-variable function $z = f(x, y)$, its ***partial derivatives*** are the derivatives obtained by considering one variable at a time, while assuming that the other is a constant, unknown parameter.

More specifically:

- The partial derivative of $z = f(x, y)$ ***with respect to x*** is the derivative of the function obtained by considering x as a variable and y as a constant:

$$\frac{\partial z}{\partial x} = z_x = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

- Similarly, the partial derivative of $z = f(x, y)$ ***with respect to y*** is the derivative of the function obtained by considering y as variable and x as a constant:

$$\frac{\partial z}{\partial y} = z_y = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

What is that strange symbol you used in the Leibniz notation?

It is a different way to write the letter d , aimed at emphasizing the fact that we are dealing with a two-variable function. That symbol is pronounced "*del.*" You will see more of it when you take second year calculus and physics courses. For now, if you prefer, you can use the shorter notation of z_x and z_y .

Example: $z = \cos xy^2 + \dfrac{e^x}{y+3}$

To compute $\dfrac{\partial z}{\partial x}$ we think of *y* as if it were an unknown constant and use appropriate rules (see if you can identify them all!) to differentiate *z* as a function of *x* only:

$$\frac{\partial z}{\partial x} = z_x = \left(-\sin xy^2\right)\left(y^2\right) + \frac{e^x}{y+3}$$

Similarly, to compute $\dfrac{\partial z}{\partial y}$ we think of *x* as if it were an unknown constant and use appropriate rules to differentiate *z* as a function of *y* only:

$$\frac{\partial z}{\partial y} = z_y = \left(-\sin xy^2\right)(2xy) + e^x \frac{-1}{(y+3)^2}$$

Are there special differentiation rules for partial derivatives?

Not for now. This is just an introduction to the topic (and an excuse to deepen your understanding of derivatives), so we shall limit our attention to those aspects that require only the usual rules of differentiation. There are some different details that arise from the chain rule when we compose a two-variable function with one of the regular kind. But you will have to wait for a more appropriate time to look at them. So, for now, just use the differentiation rules that you know.

Example: $f(x,y) = \cosh\left(x+y^2\right) + y\sin^2 x$

To compute f_x, we think of y as constant:

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\cosh\left(x+y^2\right) + y\sin^2 x\right) = \\ = \sinh\left(x+y^2\right) + 2y\sin x\cos x$$

Similarly, for f_y we think of *x* as constant:

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\cosh\left(x+y^2\right) + y\sin^2 x\right) = \\ = \left[\sinh\left(x+y^2\right)\right](2y) + \sin^2 x$$

Example: $f(x,y) = \left(x^2+3x\right)^{\sin y}$

This time, to compute f_x, we need the power rule:

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^2+3x\right)^{\sin y} = \sin y\left(x^2+3x\right)^{\sin y - 1}(2x+3)$$

But for f_y we need the general exponential rule:

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^2+3x\right)^{\sin y} = \left(x^2+3x\right)^{\sin y}\ln\left(x^2+3x\right)\times\cos y$$

And can we use these partial derivatives for something more than extra practice?

Yes: there is one interpretation of them that we can appreciate right away, although we shall not use it in this course, and a way to implement implicit differentiation by using partial derivatives. You will see both in the next section. For now, practice enough to become familiar with terminology, notation and basic computations.

Summary

- A partial derivative for a two-variable function is simply the usual derivative, computed with respect to one variable, while considering the other as a constant.

Common errors to avoid

- Just don't panic in front of partial derivatives: think of them as just usual derivatives with an additional constant that looks like a variable.

Learning questions for Section D 6-1

Review questions:

1. Describe what the partial derivatives of a two-variable function are.

2. Explain the difference between the symbols $\dfrac{dy}{dx}$ and $\dfrac{\partial y}{\partial x}$.

Memory questions:

1. What are two correct notations for the partial derivative of $z = f\left(x, y\right)$ with respect to *x*?

2. What is the proper Leibniz notation for f_x ?

Computation questions:

Compute both partial derivatives for each of the two-variable functions presented in question 1-10.

1. $f\left(x, y\right) = \dfrac{\ln xy}{y - e^{xy}}$

2. $f\left(x, y\right) = \log_y\left(x + xy\right)$

3. $f\left(x, y\right) = x^2 \sin\left(\dfrac{x}{y}\right)$.

4. $f(x,y)=\cos\left(x^2+y^2\right)$

5. $f(x,y)=\dfrac{\cosh xy}{x-\sinh xy}$.

6. $f(x,y)=x\cosh y-2\sinh xy$.

7. $f(x,y)=x\ln y-2\cos(xy)+y^2$.

8. $f(x,y)=\sin^{xy}(x+y)$.

9. $f(x,y)=y^{x+xy}$.

10. $f(x,y)=x^{x^2+y}$.

11. Determine the value of both partial derivatives of the function $f(x,y)=\dfrac{x+\cos x}{y}+\dfrac{y^2+\sin y}{x}$ at the point $(1,1)$

12. Determine the value of both partial derivatives of the function $f(x,y)=\dfrac{y-\sin x}{x}+\dfrac{x^2+\tan y}{y}$ at the point $\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$

13. Given $f(x,y)=x^3+xy^2-x^2y$, determine the values of $\dfrac{\partial f}{\partial x}(1,2)$ and $\dfrac{\partial f}{\partial y}(2,1)$.

14. Given $f(x,y)=x^{1/3}+xy^{1/2}-x^{1/2}y$, determine the values of $\dfrac{\partial f}{\partial x}(1,2)$ and $\dfrac{\partial f}{\partial y}(2,1)$.

15. Use the limit definition of partial derivative to determine $f_x(x)$ for $f(x,y)=\sqrt{xy}$.

16. Use the limit definition of partial derivative to determine $\dfrac{\partial}{\partial x}\sqrt{xy+y}$.

17. Use the limit definition of partial derivative to determine both partial derivatives of the function $z=4x^2+\sqrt{y}-2$

18. Determine $\dfrac{\partial y}{\partial x}$ if *x* and *y* are related by the equation $\sin^2 xy=\dfrac{\tanh xz}{\ln yz}$.

Theory questions:

1. Can one compute partial derivatives for a parametric function?
2. If $f(x,y)=e^{\cosh x}\ln x$, what is f_y?
3. Is it possible for a function $f(x,y)$ to be such that $f_x(a,b)$ exists, but $f_y(a,b)$ does not?
4. Convert the expression f_{xyyxx} into Leibniz notation.

Proof questions:

1. Show that if $f(x,y)=x^2-xy+\sin xy$, then $x\,f_x-y\,f_y=2x^2$
2. Show that the function $f(x,y)=e^{x^2}\sin y$ satisfies the equation $\sin y\dfrac{\partial f}{\partial x}+2x\cos y\dfrac{\partial f}{\partial y}=f$
3. Show that if $u(x,y)=y^2+xy+x^2\cos\dfrac{x}{y}$, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2u$
4. Show that for the function $f(x,y)=e^{x^2+y^2}$ the identity $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=2f\ln f$ is satisfied.

Templated questions:

1. Construct a reasonably simple function of two variables and compute its partial derivatives.

What questions do you have for your instructor?