Chapter 6: Derivatives and other types of functions — **Section 3**

Higher order partial derivatives

What you need to know already:	What you can learn here:
➤ What partial derivatives are and how to compute them.	➤ The notation and some basic important facts about higher order partial derivatives.

We have seen that a partial derivative is just a regular derivative, but computed on a two-variable function by considering the other variable as constant. That means that we can consider higher derivatives with respect to one of the variables, just like we did for usual functions. All we need to is to add is a minor change of notation to point out that we are dealing with a partial derivative.

Definition

If $z = f(x, y)$ is a two-variable function that is differentiable twice with respect to *x*, then its ***second partial derivative for x*** is denoted by:

$$\frac{\partial^2 f(x, y)}{\partial x^2} = f_{xx} = z_{xx}$$

Similarly, the ***second partial derivative with respect to y***, if it exists, is denoted by:

$$\frac{\partial^2 f(x, y)}{\partial y^2} = f_{yy} = z_{yy}$$

Example: $z = y^2 \ln x$

Here are the two partial derivatives of this function:

$$z_x = y^2 \frac{1}{x} = \frac{y^2}{x} \quad ; \quad z_y = 2y \ln x$$

Therefore, the second partial derivatives with respect to *x* and *y* are, respectively:

$$z_{xx} = \frac{\partial}{\partial x}\left(\frac{y^2}{x}\right) = -\frac{y^2}{x^2} \quad ; \quad z_{yy} = \frac{\partial}{\partial y}(2y \ln x) = 2\ln x$$

But these partial derivatives are still functions of two variables, so do we have to keep differentiating them always with respect to the same variable?

Clearly there is no need for that! There is no reason why we cannot alternate variable and differentiate first with respect to one variable, then to the other. Again, all we need to do is be careful about the notation.

Definition

The ***second mixed partial derivatives*** of a two-variable function $z = f(x, y)$ are denoted by:

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x, y)\right) = f_{xy} = z_{xy}$$

$$\frac{\partial^2 f(x, y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x, y)\right) = f_{yx} = z_{yx}$$

Warning bells

Notice and remember that the order of differentiation is indicated in the subscript ***from right to left***, just as in the composition of functions, so that, for instance:

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x, y)\right)$$

Example: $z = y^2 \ln x$

We have seen earlier that here $z_x = \frac{y^2}{x}$ and $z_y = 2y \ln x$.

Therefore, the second mixed partial derivatives are, respectively:

$$z_{yx} = \frac{\partial z_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y^2}{x}\right) = \frac{2y}{x} \quad ; \quad z_{xy} = \frac{\partial z_y}{\partial x} = \frac{\partial}{\partial x}(2y \ln x) = \frac{2y}{x}$$

You picked an interesting example: in this case they are the same!

Am I smart or what! Actually, it's "*what*", since this is not a coincidence or a special case at all.

Technical fact

If $z = f(x, y)$ is a two-variable function, whose ***second mixed partial derivatives are continuous***, then they are ***equal***, regardless of the order of differentiation:

$$f_{xy} = f_{yx}$$

This means that the second mixed partials are different only in very rare cases, in exceptions that must be constructed with some care and viciousness! For our purposes, we shall just ignore them. Moreover, I will omit the proof of this fact, since it consists of an uninspiring sequence of technical steps and checks. Instead, here is another example, after which you can try to compute these mixed higher derivatives by yourself and check that they are equal.

Example: $f(x, y) = \sin x^2 y$

We start from the first partials:

$$f_x = 2xy \cos x^2 y \quad ; \quad f_y = x^2 \cos x^2 y$$

Then we compute the two pure partials:

$$f_{xx} = (2xy)_x \cos x^2 y + 2xy\left(\cos x^2 y\right)_x$$
$$= 2y \cos x^2 y - 4x^2 y^2 \sin x^2 y$$
$$f_{yy} = x^2\left(\cos x^2 y\right)_y = -x^4 \sin x^2 y$$

Finally we compute the two mixed partials and check that they are the same:

$$f_{yx} = \frac{\partial}{\partial y}(2xy)\cos x^2 y + 2xy\frac{\partial}{\partial y}\left(\cos x^2 y\right)$$
$$= 2x\cos x^2 y - 2x^3 y\sin x^2 y$$
$$f_{xy} = \frac{\partial x^2}{\partial x}\cos x^2 y + x^2\frac{\partial}{\partial x}\left(\cos x^2 y\right)$$
$$= 2x\cos x^2 y - 2x^3 y\sin x^2 y$$

They are indeed equal ☺.

What about even higher order derivatives?

You are ambitious today! Well, they are done in the obvious way.

Knot on your finger

Higher partial derivatives can be computed to any order, provided the required derivatives exist.

The order of differentiation in the notation for a higher mixed partial is indicated ***from right to left***, but it does not matter, provided such mixed partial is continuous.

Example: $f(x,y) = \sin x^2 y$

Let us go one step further with this function and compute f_{xyy} :

$$f_{xyy} = \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial y^2}\sin x^2 y\right) = \frac{\partial}{\partial x}\left(-x^4\sin x^2 y\right)$$
$$= -4x^3\sin x^2 y - \ 2yx^5\cos x^2 y$$

But we can also compute f_{yyx} and we should get the same function, right? Let's see:

$$f_{yyx} = \frac{\partial^2}{\partial y\partial y}\left(2xy\cos x^2 y\right) = \frac{\partial}{\partial y}\left(2x\cos x^2 y - 2x^3 y\sin x^2 y\right)$$
$$= -2x^3\sin x^2 y - 2x^3\sin x^2 y - 2yx^5\cos x^2 y$$
$$= -4x^3\sin x^2 y - 2yx^5\cos x^2 y$$

Yes, as expected. I leave you the fun of checking that the third possible mixed partial f_{yxy} is also the same.

Warning bells

The order of differentiation for mixed partial derivatives does not matter, but only provided that:

- such mixed partial is continuous AND
- the number of times each variable is used as the variable of differentiation is the same.

Therefore, while we can assume that

$$f_{xxy} = f_{xyx} = f_{yxx}$$

in general it is NOT TRUE that $f_{xxy} = f_{xyy}$ since we are not using each variable the same number of times in the two cases.

Well, that makes sense! If we take different derivatives, we cannot expect the same conclusion.

Obviously I agree, but I have seen students make mistakes in this way, so I hope you will not join them.

Summary

- Higher order partial derivatives can be computed just as for usual derivatives.
- Higher partial derivatives may be computed with respect to a single variable, or changing variable at each successive step, so as to obtain a mixed partial derivative.
- If a mixed partial derivative is continuous, the order in which the variables were used in the computation is irrelevant, as long as it is done the required number of times for each variable.

Common errors to avoid

- Keep track of the algebra and of which variable you are using at each step. Otherwise, you may end up with a derivative different from what you want.

Learning questions for Section D 6-3

Review questions:

1. Describe the notation used for higher order partial derivatives.
2. Explain how the order of differentiation in mixed partial derivatives is indicated in its notation.

Memory questions:

1. What is the correct Leibniz notation for all second derivatives of a two-variable function $z = f(x, y)$?
2. What is the correct short notation for all second derivatives of a two-variable function $z = f(x, y)$?
3. In which order are the variables indicated in a mixed partial derivative?
4. How do we call a higher derivative of a two-variable function for which each variable is used at some stage as the variable of differentiation?
5. When are the mixed partial derivatives of a function of two variables independent of the order of differentiation?

Computation questions:

Compute all second partial derivatives for the functions provided in questions 1-6.

1. $f(x,y)=y^{x^2}$.

2. $z=x^2-xy+y^2$

3. $z=\dfrac{x}{y}+e^x\sin y$

4. $z=f(x,y)=x^2-\dfrac{x}{y}+y^2$.

5. $z=\dfrac{x^4}{2y}+e^{2x}\sin y$

6. $u(x,y)=x^3y^2+4x^2\cos y$

7. Compute f_{xyx} for $f(x,y)=\dfrac{x^2y}{y-\sinh y}$.

8. Compute f_{xxy} for the function defined by $f(x,y)=\ln xy-2\cos(x-y)+xy^2$.

9. Determine $\dfrac{\partial^3 z}{\partial x\partial y\partial x}$ for $z=\dfrac{x\cos y^y}{3}-x^2e^{\cosh y}$. Be smart in your use of properties of partial derivatives ☺

10. Given the function $f(x,y)=e^x\cos y$, determine the formula for $\left(\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}\right)\left(\dfrac{\partial^2 f}{\partial x\partial y}\right)$.

11. Given the function $f(x,y)=e^{x^2}\sinh y$, determine the formula for $\left(\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}\right)^2-\dfrac{\partial^2 f}{\partial x\partial y}$.

Theory questions:

1. Is it always correct to switch the order of differentiation for mixed partial derivatives?

2. By visual inspection only, what is $\dfrac{\partial^6\left(x^2y^2\right)}{\partial x^4\partial y^2}$?

Templated questions:

1. Construct a reasonably simple two-variable function and compute all its second partial derivatives.

What questions do you have for your instructor?