Chapter 6: Derivatives and other types of functions *Section 4*

Slope and tangent lines for parametric curves

What you need to know already:	***What you can learn here:***
➤ The connection between derivatives and slopes; how to compute derivatives.	➤ How to use derivatives to compute slopes and tangent lines for parametric curves.

Even though parametric curves are defined in a way different from the usual function approach, they are still curves. As such, it makes sense to try to compute their slope and their tangent line at any of their point. There is one obvious way to do that when the parameter can be eliminated.

Strategy for computing the slope of a parametric curve if the parameter can be eliminated

If the formulae for a parametric curve $(x(t), y(t))$ can be manipulated so as to eliminate the parameter and obtain an equation of the form $f(x, y) = 0$, then its slopes and tangent lines can be obtained by using regular or implicit differentiation.

But this is old hat and you should respond to it with a "*doh*!" The more interesting question is what to do when the parameter cannot be eliminated, so that we are stuck with the original formulae. Can we compute the slope then?

I suspect so, or you would not bring it up!

Technical fact

The slope of any parametric curve of the form $(x(t), y(t))$ is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

Proof

It may seem that, to convince ourselves of the truth of this fact, all we have to do is divide top and bottom of the first fraction by *dt*. That is, in fact , the soul of the real proof, but remember that the first fraction is not a real fraction, but the limit of one; it is really just a symbol for the derivative.

However, a complete proof of this fact only requires our checking a number of details related to the limit that defines a derivative. It is another instance of a complex and uninformative proof: we trust the many mathematicians who have checked on our behalf!

Example: $\left(\frac{e^t \cos t}{t+2}, \ln \cosh\left(t^2 - 3t\right) \right)$

Attempting to eliminate the parameter for this curve would be a time-wasting quest. Instead, we use the parametric formula:

$$\frac{dy}{dx} = \frac{\left[\ln \cosh\left(t^2 - 3t\right)\right]'}{\left(\frac{e^t \cos t}{t+2}\right)'}$$

Not totally simple, but it can be done:

$$\frac{dy}{dx} = \frac{\frac{1}{\cosh\left(t^2 - 3t\right)}\left[\sinh\left(t^2 - 3t\right)\right](2t-3)}{\frac{\left(e^t \cos t - e^t \sin t\right)(t+2) - e^t \cos t}{(t+2)^2}}$$

And all we need now is the value of t for which we need to compute it. Once we know that, the rest is routine.

Now that we know how to compute the slope of a parametric curve, we can compute its tangent lines as well, by using the standard formula:

$$y = f'(c)(x-c) + f(c)$$

Example: $\left(\frac{e^t \cos t}{t+2}, \ln \cosh\left(t^2 - 3t\right) \right)$ *at* $\left(\frac{1}{2}, 0 \right)$

We have just obtained the formula for the slope of this curve at any point, so we just need to apply it here. But what is the value of t that determines the given point? We start from the second coordinate, which seems less complicated. We need:

$$\ln \cosh\left(t^2 - 3t\right) = 0 \quad \Rightarrow \quad \cosh\left(t^2 - 3t\right) = 1$$

$$\Rightarrow \quad t^2 - 3t = 0 \quad \Rightarrow \quad t = 0, 3$$

For $t = 0$ the first coordinate becomes:

$$x(0) = \frac{e^0 \cos 0}{0+2} = \frac{1}{2}$$

So, it works. But perhaps the second works as well: remember that a parametric curve can go through the same point more than once. We check:

$$x(3) = \frac{e^3 \cos 3}{3+2} \approx \frac{20(-1)}{5} \approx -4$$

Not even close! For $t = 0$ the slope is given by:

$$\frac{dy}{dx} = \frac{\frac{1}{\cosh 0}\left[\sinh 0\right](0-3)}{\frac{\left(e^0 \cos 0 - e^0 \sin 0\right)(0+2) - e^0 \cos 0}{(0+2)^2}} = \frac{0}{\frac{2-1}{4}} = 0$$

Therefore the tangent line is just $y = 0$

Will the number crunching always be so easy?

In real life, no. But in this course, yes, especially when the formula is so complicated!

What about higher derivatives? Can they also be done?

Yes, but as you can imagine, they become more and more complicated. So, I will just show you the formula for the second derivative, since we shall use it later when trying to graph a parametric curve.

Technical fact

The concavity of any parametric curve of the form $\left(x(t), y(t)\right)$ is given by the formula:

$$\frac{d^2y}{dx^2} = \frac{y''x' - x''y'}{(x')^3}$$

Here the primes indicate derivatives with respect to ***t***.

What a mess: where does it come from?

From the meaning of second derivative and the formula for the slope of a parametric curve. Here is the proof.

Proof

We are looking for the derivative of the slope, which we now know to be:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'}{x'}\right)$$

But notice that we are looking for the derivative with respect to *x* of a fraction written in terms of *t*! Well, consider the new parametric curve:

$$\left(x(t), \frac{y'(t)}{x'(t)}\right)$$

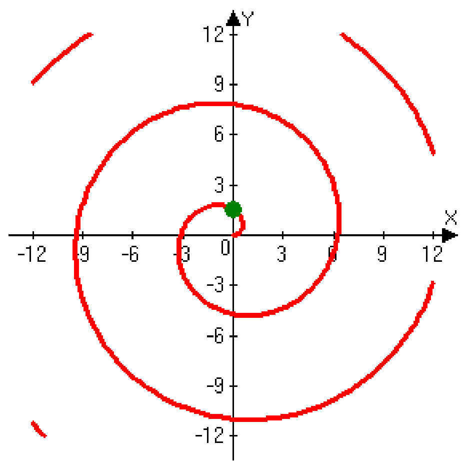
The second derivative we are looking for is actually the slope of this curve. We can therefore use again the formula for the slope of a parametric curve and obtain:

$$\frac{d^2y}{dx^2} = \frac{d(\dot{y}/\dot{x})/dt}{dx/dt}$$

For the numerator, we can use the quotient rule, thus obtaining:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{y''x' - x''y'}{(x')^2}}{x'} = \frac{y''x' - x''y'}{(x')^3}$$

Example: $\left(t\cos t, t\sin t\right), t \geq 0$



This is a spiral, whose slope and concavity we can compute at any point. For instance, for $t = \frac{\pi}{2}$ we are at the point of coordinates $\left(0, \frac{\pi}{2}\right)$, which is shown in the picture. To find the slope there we compute:

$$x' = \cos t - t\sin t$$
$$y' = \sin t + t\cos t$$

Therefore:

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{\sin t + t\cos t}{\cos t - t\sin t}$$

At our point the slope is:

$$\frac{dy}{dx}=\frac{\sin\frac{\pi}{2}+\frac{\pi}{2}\cos\frac{\pi}{2}}{\cos\frac{\pi}{2}-\frac{\pi}{2}\sin\frac{\pi}{2}}=\frac{1}{-\frac{\pi}{2}}=-\frac{2}{\pi}\approx-\frac{2}{3}$$

This is consistent with what we see in the graph: the slope is going down with a slope that we can visually estimate to be close to -2/3.

For the second derivative, we get:

$$x''=-\sin t-\sin t-t\cos t=-\left(2\sin t+t\cos t\right)$$
$$y''=\cos t+\cos t-t\sin t=2\cos t-t\sin t$$

Therefore the concavity is given by:

$$\frac{d^2y}{dx^2}=\frac{y''x'-x''y'}{\left(x'\right)^3}$$
$$=\frac{\left(2\cos t-t\sin t\right)\left(\cos t-t\sin t\right)+\left(2\sin t+t\cos t\right)\left(\sin t+t\cos t\right)}{\left(\cos t-t\sin t\right)^3}$$

At our point this becomes:

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}}=\frac{\left(-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right)+\left(2\right)\left(1\right)}{\left(-\frac{\pi}{2}\right)^3}=\frac{\frac{\pi^2}{4}+2}{-\frac{\pi^3}{8}}=-\frac{2\pi^2+16}{\pi^3}$$

We shall see later that this negative value is consistent with the fact that at our point the curve is concave down.

I seem to remember that, but I guess we have to wait until we study graphs in more details before all this will become clearer.

Certainly focussing on graphs will help; for now work on using the formulae and interpreting the slope values.

Summary

- The slope formula for a parametric curve can be found with the usual methods when the parameter can be eliminated.
- When the parameter cannot be eliminated, it can be computed directly as the ratio of the derivatives of the two functions: $\frac{dy}{dx}=\frac{y'}{x'}$ with respect to *t*.
- The second derivative formula can be computed, but it becomes more complex and can only be applied later, in relation to graphs.

Common errors to avoid

- Don't panic if the formula you obtain becomes complicated: in that case you will not be asked to reduce it and you will need to evaluate it only at easy values.
- Remember that the general formula for the tangent line is still the same, even if the curve is given in parametric form.

Learning questions for Section D 6-4

Review questions:

1. Explain how to construct the equation of the line tangent to a parametric curve at a given point.

Memory questions:

1. What is the formula that provides the slope of a parametric curve $\left(x(t),\ y(t)\right)$?

2. What is the formula for the equation of a line tangent to a parametric curve $\left(x(t),\ y(t)\right)$ at a given point $\left(a, b\right)$?

Computation questions:

In questions 1-10, construct the equation of the line tangent to the given parametric curve at the given point.

1. $\left(2^t, 3\left(4^t\right)\right)$ at $\left(2, 12\right)$.

2. $\left(e^{\cos t}, e^{\sqrt{t}}\right)$ at $t = \pi$

3. $\left(\tan t, 2\cos t\right)$ at $t = \dfrac{\pi}{4}$

4. $\left(\sin 2t,\ 2\cos t\right)$ at $\left(0, 2\right)$

5. $\left(\ln t^2, \ln^2 t\right)$ at $t = e$

6. $\left(t^2 + \ln t,\ t^2 - \ln t\right)$ at $\left(1, 1\right)$

7. $\left(t^2 + \dfrac{1}{t}, 1 - t^2\right)$ at $\left(\dfrac{9}{2}, -3\right)$

8. $\left(t^2 + \sqrt{t}, 1 - \sqrt{t}\right)$ at $\left(18, -1\right)$

9. $\left(e^t, e^{3t}\right)$ at $t = 2$.

10. $\left(2e^{2t}, \dfrac{1}{3}e^{3t}\right)$ at $t = 1$.

11. Identify all points on the curve $\left(\sqrt{t^2-1}, \sqrt{t^2+1}\right)$, if any, where the tangent line has slope 3.

12. Find the points on the parametric curve $\left(3t^2+1,\ 4t^3-2t+2\right)$ where the tangent line is perpendicular to the line $y = 3x$

13. At which point(s) of the curve $\left(t^3, t^2-6t\right)$ is the tangent line parallel to the bisectrix of the second quadrant?

14. Determine the equations of all lines tangent to the curve $\left(t^2-2t, 2t^3-3t^2\right)$ and forming an angle of $\pi/3$ with the x-axis.

15. Determine the coordinates of the points on the parametric curve $\left(t^2, \sin t\right)$ where the tangent line is horizontal.

16. Determine the equation of one of the lines tangent to the curve $\left(t^2-2t, 2t^3-3t^2\right)$ and forming an angle of $\pi/4$ with the x-axis.

Theory questions:

1. Which derivative is 0 at the points of a parametric curve where the tangent line is vertical?

2. What can we say about the slope of a parametric curve at values of t for which $x'(t) = 0$?

3. For a parametric curve, does *dy/dt* provide the slope, the orientation, both or neither?

4. In a parametric curve is it possible that y is an increasing function of t and yet the graph is always decreasing?

Templated questions:

1. Construct a reasonably simple parametric curve and then compute its slope formula and the equation of the tangent line at one of its points.

2. For any of the parametric curves provided in *Computation questions* 1-10, compute the second derivative at the given point.

What questions do you have for your instructor?