Chapter 6: Derivatives and other types of functions — *Section 5*

Slopes and tangents for polar curves

What you need to know already:	*What you can learn here:*
➤ Definition and basic properties of polar curves. ➤ How to construct the equation of the tangent line for a parametric curve.	➤ How to compute the slope of a polar curve and the special property of such curves at the pole.

Since any polar curve can be written as a parametric curve, all the calculus methods that apply to parametric curves can also be used for polar curves. In particular, we can use what was discussed in the previous section to obtain the equation of a line tangent to a polar curve.

Since the parametric version of a polar curve uses the rather involved formulae

$$(x, y) = \left(r(\theta)\cos\theta, r(\theta)\sin\theta\right)$$

we should expect the corresponding slope formula to also be complicated.

Technical fact

The slope of a polar curve $r = r(\theta)$ is given by the formula:

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

Proof

This is a standard application of the slope formula for a parametric curve. Since, in general:

$$\left(x(t), y(t)\right) \Rightarrow \frac{dy}{dx} = \frac{y'}{x'}$$

For a polar curve we have:

$$\left(r\cos\theta, r\sin\theta\right) \Rightarrow \frac{dy}{dx} = \frac{\left(r\sin\theta\right)'}{\left(r\cos\theta\right)'}$$

Since r is a function of θ, we use the product rule on both numerator and denominator, thus obtaining the claimed formula.

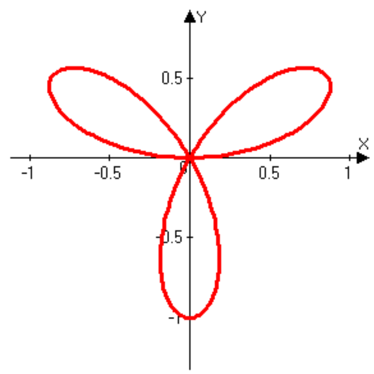
I am getting confused between polar and Cartesian and why isn't the slope simply given by $\frac{dr}{d\theta}$ *?*

Excellent question, so let me clarify it somewhat.

Warning bells

The ***slope*** of a curve refers ***always*** to the rate of change of the *y*-coordinate with respect to the *x*-coordinate, regardless of how the curve is described and/or which coordinate system is used.

Example: $r = \sin 3\theta$

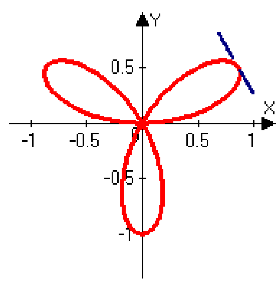


This is the so-called 3-petal rose, even though it does not look like a rose and has, in fact, 6 overlapping petals!

To compute its slope formula, we apply the given standard formula:

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}$$

Not a pretty formula, but, with a little patience, we can evaluate it at any point. For instance, for $\theta = \dfrac{\pi}{6}$ we are at the point shown here together with its tangent line.



Its polar and Cartesian coordinates are, respectively:

$$(r,\theta) = \left(1, \frac{\pi}{6}\right) \qquad (x, y) = \left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

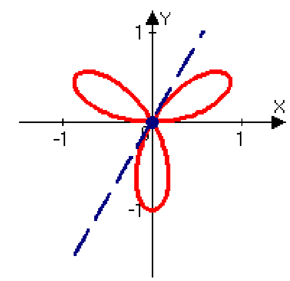
It looks like the tangent line at this point should go downward; hence its slope should be negative. We use the formula to confirm:

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{3\cos\frac{\pi}{2}\sin\frac{\pi}{6} + \sin\frac{\pi}{2}\cos\frac{\pi}{6}}{3\cos\frac{\pi}{2}\cos\frac{\pi}{6} - \sin\frac{\pi}{2}\sin\frac{\pi}{6}} = \frac{0 + \frac{\sqrt{3}}{2}}{0 - \frac{1}{2}} = -\sqrt{3}$$

Just as expected. Therefore, the equation of the tangent line is

$$y = -\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right) + \frac{1}{2}:$$

Similarly, at $\theta = \dfrac{\pi}{3}$ we get $r = \sin\pi = 0$.



We are at the pole, with a tangent of slope:

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{3\cos\pi\sin\frac{\pi}{3} + \sin\pi\cos\frac{\pi}{3}}{3\cos\pi\cos\frac{\pi}{3} - \sin\pi\sin\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \tan\frac{\pi}{3} = \sqrt{3}$$

The equation of the tangent line is therefore $y = \sqrt{3}x$.

It looks like there is a lot of need for a calculator here!

Generally, yes, but the examples and questions you will see will be designed so that the calculations can be done easily and quickly, most times without a calculator.

Moreover, there is a special feature that turns out to be quite useful, both theoretically and for computational purposes.

Recall that in the polar system of coordinates, the key role is played by the pole, so let us analyze the slope with which a polar curve goes through the pole.

The formula for the slope involves *r*, and if we are at the pole, $r = 0$. This makes the computations simpler.

Technical fact

The slope of a polar curve $r = r(\theta)$ at the pole is:

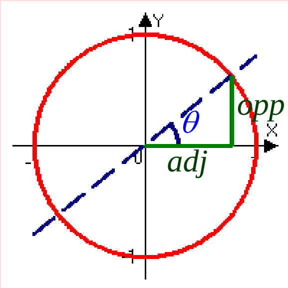
$$\left.\frac{dy}{dx}\right|_{r=0} = \frac{r'\sin\theta}{r'\cos\theta} = \tan\theta$$

This means that if a polar curve ***crosses the pole*** for a certain value $\theta = \alpha$, its tangent line there forms ***an angle of*** α with the positive *x*-axis.

Proof

All we need to notice is that, by definition, $\tan\theta$ is obtained by computing opposite over adjacent for a line that forms an angle of θ with the positive *x* axis. But opposite over adjacent is the same as rise over run, which is the slope of that line, exactly what the derivative provides:

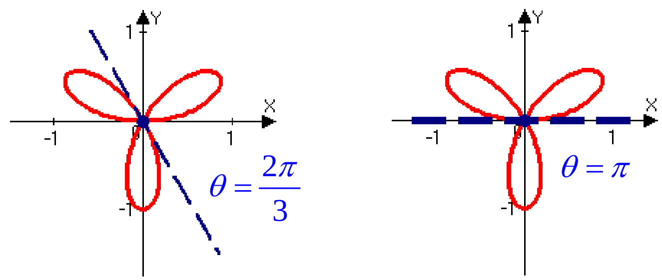
$$m = \frac{rise}{run} = \frac{opp}{adj} = \tan\theta$$



Example: $r = \sin 3\theta$

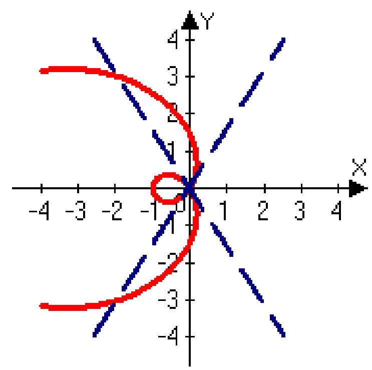
We have just seen that this curve goes through the pole when $\theta = \frac{\pi}{3}$ and that the tangent line there has slope $\tan\frac{\pi}{3}$. Well, this was not a coincidence!

This curve also goes through the pole whenever $\sin 3\theta = 0$, so in particular when $\theta = \frac{2\pi}{3}$ or $\theta = \pi$. The tangent lines there will form those same angles with the positive *x* axis, as we can see in these graphs.



Example: $r = \theta^2 - 1$

This polar curve goes through the pole when $\theta^2 = 1$, that is, when $\theta = \pm 1$. But this is approximately $\theta \approx \pm\pi/3$. Therefore, the slope there will be, respectively, $\tan\theta = \tan(\pm 1) = \pm\tan 1$, which, still by the way, is approximately $\pm\sqrt{3}$.



Summary

- The slope of a polar curve can be computed by using its parametric form.
- The general formula for a parametric curve is not simple, but not too complicated either.
- The slope of a polar curve at the pole is the trigonometric tangent of the angle that the geometric tangent forms there with the positive *x* axis. (You may want to read that twice to make sure you get the technical meaning!)

Common errors to avoid

- Watch the computational steps and make sure the values of the slope you obtain are reasonable. A graphing calculator may help you with that.

Learning questions for Section D 6-5

Review questions:

1. Describe how we obtain the formula for the slope of a polar curve.

2. Explain why the slope of a polar curve when it crosses the pole equals the trigonometric tangent of the value of θ for which it crosses it.

Memory questions:

1. Which formula provides the slope of a polar curve $r = f(\theta)$?

2. What is the formula for the slope of line tangent to a polar curve at $(0, \theta)$?

Computation questions:

1. Which formula provides the slope of the curve $r = e^{-\theta}$?

2. Find the slope of the tangent lines at the pole for the polar curve $r = 3\dfrac{\cos 2\theta}{\cos\theta}$.

3. What is the Cartesian equation of the line tangent to the polar $r = \theta^2 - \dfrac{8}{\theta}$ at the pole?

4. What is the Cartesian equation of the line tangent to the polar $r=\theta^3-\frac{27}{\theta}$ at the pole?

5. Find the slope of the line tangent to the polar curve $r\sqrt{\theta}=1$ at $\theta=\frac{\pi}{4}$.

6. Determine the equations of all lines that are tangent to the polar curve $r=3-3\cos\theta$ and horizontal.

7. Find the points on the polar curve $r=1-2\cos\theta$ where the tangent line is horizontal and those where it is vertical.

8. Find the points on the polar curve $r=3-2\sin\theta$ where the tangent line is horizontal and those where it is vertical.

9. Determine the points of intersection of the polar curves $r=1+\sin\theta$ and $r=3\sin\theta$ and identify the lines tangent to the first curve at each of these points. (Use your calculator to make sure you don't miss any point, but justify all of them algebraically).

10. Identify the values of θ for which the polar curve $r=2+4\sin\theta$ has horizontal tangent lines and find the slope of the tangent lines at the pole.

11. Which equation in θ will allow us to identify the points where this cochleoid $r=\frac{\sin\theta}{\theta}$ has a vertical tangent line?

12. Determine the angle with which the polar curve $r=\theta\sin\theta, 0\le\theta\le 2\pi$ intersects the *y*-axis, including when it intersects at the pole.

13. Determine which lines are tangent to the polar curve $r=\theta+\theta\sin\theta$ at the origin.

Theory questions:

1. If a polar curve $r=f(\theta)$ is such that $f\left(\frac{\pi}{4}\right)=0$, what is the slope of the tangent to this curve at $\theta=\pi/4$?

2. If a polar curve reaches the pole for $\theta=\pi/6$, what is the slope of the tangent line there?

3. If a polar curve $r=f(\theta)$ is such that $f(3)=0$, what is the slope of the curve at the corresponding point?

4. What must equal 0 in order for the tangent line to a polar curve to be horizontal?

5. What must equal 0 in order for the tangent line to a polar curve to be vertical?

6. If a polar curve goes through the pole for $\theta=c$, what is the equation of its tangent line there?

7. If a polar curve goes through the pole only if $\theta=\frac{\pi}{3}+n\pi$, can it have a minimum there?

Proof questions:

1. Construct the formula for the concavity of a polar curve $r = r(\theta)$.

Templated questions:

1. Construct a reasonably simple polar curve and, from it, construct the formula that provides its slope.
2. Construct a reasonably simple polar curve and find the slope with which it crosses the pole.

What questions do you have for your instructor?