Chapter 7: Linear approximations — *Section 2*

Error estimates

What you need to know already:

- All basic facts and methods about derivatives and differentials.
- The method of linearization.

What you can learn here:

- How to use the method of linearization to estimate errors in measurement.

In the last section we saw how to use the tangent line as a tool to estimate the values of a complicated function by doing simpler operation at suitable values of x. The same underlying idea allows us to develop a similar method of estimation, but this time for errors. Let me recap.

If a function $f(x)$ is differentiable at $x=c$, then:

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

Therefore, for small values of h:

$$f'(c)\approx\frac{f(c+h)-f(c)}{h}$$

If we multiply both sides by h we arrive at the following conclusion.

Technical fact

If $f(x)$ is a function that is differentiable at $x=c$ and h is a small number, then:

$$f'(c)\,h\approx f(c+h)-f(c)$$

The interesting part is how this approximate equality can be interpreted.

Technical fact

If h is interpreted as a ***small error*** – or ***variation*** – in the measurement of the x variable, then the quantity $f(c+h)-f(c)$ can be interpreted as the ***corresponding error*** in the measurement of the y variable:

$$h=\Delta x \quad \leftrightarrow \quad f(c+h)-f(c)=\Delta y$$

Such related error is well ***approximated*** by the differential:

$$\Delta y=f(c+h)-f(c)\approx f'(c)\,h=dy$$

But we knew that already from what we learned about increments and differentials!

I am glad that you recognize this; you are correct, since all we are adding now is the interpretation! If we consider those increments as errors in measurement, this approximation tells how to estimate the error in the dependent variable by using differentials. Time for examples.

Example: $A = s^2, \quad A' = 2s$

The first formula tells us how to compute the area of a square based on the measurement of its side. So, if the side is 4 *cm* long, the area is:

$$A = 4^2 = 16$$

Simple enough. But what if our measurement of the side was off by, say, 0.2 *cm*? How much off would we be in the measurement of the area? One way to answer this question is to compute the corresponding increment:

$$\Delta A = A(4.2) - A(4) = 17.64 - 16 = 1.64\ cm^2$$

But our interpretation of differentials tells us that we can estimate the same quantity by using *dy* instead:

$$dA = A'(4)\Delta s = 2(4)(0.2) = 1.6\ cm^2$$

In this way we have obtained a very similar estimate, and that is the point of the method.

But in this case computing the increment/error was not that difficult.

True, but in some other situations evaluating the function may be much more complicated and computationally intensive, while computing and evaluating the derivative at a suitable value may be easier.

Knot on your finger

Estimating an error by using differentials instead of increments is an ***alternative*** and ***approximate*** method. As such, sometimes it is efficient, but sometimes it is not.

Example: 

The end of a cylindrical log has a radius measured at 40 *cm*, but the measurement may be off with a possible error of 0.3 *cm*. What is the corresponding error in the area?

Since the area we are considering is that of a circle, we can use the formula:

$$A = \pi r^2$$

The actual error when the measurement is excessive is:

$$\Delta A = A(40.3) - A(40) = \pi\left[(40.3)^2 - 40^2\right] = 24.09\pi \approx 75.68\ cm^2$$

But maybe the error is because the measurement was too short, in which case we have:

$$A(40) - A(39.7) = \pi\left[40^2 - (39.7)^2\right] = 23.91\pi \approx 75.11\ cm^2$$

These computations are still doable, but more awkward. Notice also that we need to consider two cases. If we take the differential approach, we use:

$$A(r) = \pi r^2 \quad \Rightarrow \quad A'(r) = 2\pi r$$

Therefore:

$$dA = A'(40)\Delta r = 2\pi 40\frac{3}{10} = 24\pi \approx 75.4\ cm^2$$

Arguably simpler and certainly one computation only. What we lose is the accuracy, but we may be close enough for our practical purposes.

There is one more useful step we can take. When we looked at the linearization of a function, we looked briefly at the issue of how small h can be and how good is the estimate we compute. We concluded that the answer to this question depends on the context.

When dealing with errors, we can say something more precise: we can compute what percentage of the actual measurement is due to the error.

Definition

The ***relative error*** of a measurement is the ratio between the size of the error and the expected size of the measurement:

$$\text{Relative error} = \left|\frac{\Delta y}{y}\right| = \left|\frac{f(c+h)-f(c)}{f(c)}\right|$$

The ***percentage error*** of a measurement is the relative error times 100:

$$\text{Percentage error} = \left|\frac{\Delta y}{y}\right| \times 100\%$$

This is probably not new to you from other courses, but here is how calculus comes in.

Technical fact

The relative and percentage errors may be ***approximated*** by, respectively:

$$\text{Relative error} \approx \left|\frac{dy}{y}\right| = \left|\frac{f'(c)\,dx}{f(c)}\right|$$

$$\text{Percentage error} \approx \left|\frac{f'(c)\,dx}{f(c)}\right| \times 100\%$$

Knot on your finger

In some cases, these formulae may be simplified algebraically, thus leading to an easier procedure.

Both quantities provide a ***relative size*** for the error, thus allowing us to decide whether such error is ***within acceptable margins***.

Example: $A = s^2$

In our computation of the area based on the length of the side we obtained an error of $1.6\ cm^2$ for an expected measurement of $16\ cm^2$. Therefore, the relative error was:

$$Rel\ err = \frac{1.6}{16} = 0.1 \quad \leftrightarrow \quad 10\%$$

Notice also that, for our function

$$Rel\ err = \frac{(2s)\,ds}{s^2} = \frac{2ds}{s}$$

Since the assumed error in the length is $ds = 0.2$, we get:

$$Rel\ err = \frac{2(0.2)}{4} = 0.1 \quad \leftrightarrow \quad 10\%$$

Same result easier computation.

Is a 10% error too much? Are you prepared to accept it? Well, that still depends, but at least we have an assessment based on the specific case.

Example:

In the cylindrical log example, we computed an estimated error of $24\pi\ cm^2$ for an expected measurement of $\pi 40^2 = 1600\pi\ cm^2$. Therefore the relative and percentage errors are:

$$Rel\ err = \frac{24\pi}{1600\pi} = 0.015 \quad \leftrightarrow \quad 1.5\%$$

Again, working on the formula first we get:

$$Rel\ err = \frac{2\pi r dr}{\pi r^2} = \frac{2dr}{r}$$

This, with our numbers, becomes:

$$Rel\ err = \frac{2(0.3)}{40} = \frac{0.6}{40} = 0.015$$

Again, same conclusion, but easier computations, especially since π can now be cancelled and does not contribute to further approximations.

And is 1.5% acceptable? It still depends on the context and the needs, but at least now we have a measure with which to work.

Summary

- Differentials and the related linear approximations may be used to estimate errors in measurement.
- Relative errors and percentage errors can also be estimated in this way and provide an assessment of the size of the error in relation to the quantity being measured.

Common errors to avoid

- Don't get lost in the terminology, the notation, or the meaning of these concepts. They are simple AFTER you have clarified them, but you do need to clarify them through exploration and practice work.

Learning questions for Section D 7-2

Review questions:

1. Describe how differentials may be used to estimate errors and relative errors.

1. Discuss the advantages and disadvantages of using differentials to estimate errors and relative errors.

Memory questions:

1. Which formula uses the differential to estimate an error?

2. What is the most common applied interpretation of differentials?

3. Which formula defines the linear approximation of a relative error in measurement for a function $y = f(x)$?

Computation questions:

1. By using a calculator, compute the relative and percentage errors in estimating the value of $\cos\frac{3}{2}$ by using $\cos\frac{\pi}{2}$ instead. Then use differentials for the function $y = \cos x$ to estimate the same errors. Also explain why the values you obtain are so large.

2. By using a calculator, compute the relative and percentage error in estimating the value of $\sin\frac{3}{2}$ by using $\sin\frac{\pi}{2}$ instead. Then use differentials for the function $y = \sin x$ to estimate the same errors.

Theory questions:

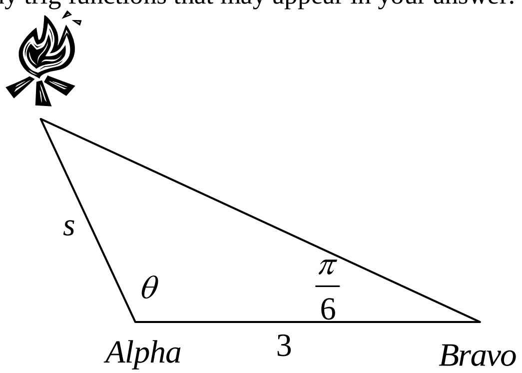
1. What is a relative error?

2. Why are relative errors also reported as percentages?

Application questions:

1. The head of a bolt is in the shape of a regular hexagon. If its side is supposed to be 0.8cm, but is measured with a possible error or 0.1mm, what is the relative error in the estimated area of the bolt's head, as computed by using differentials?

2. Alpha and Bravo are the code names of two fire monitoring stations located 3 km apart along the E-W township road carved at the edge of a large meadow, as depicted in the graph. At some point station Bravo locates a fire at bearing N60°W, but station Alpha is having some technical difficulties and can locate the fire at bearing N45°W only within an error of ±3°. Use differentials to estimate the corresponding error in the distance from Alpha to the fire. No need to evaluate any trig functions that may appear in your answer.



3. The period of a pendulum's motion is related to the pendulum's length through the equation. $T = 2.01\sqrt{L}$, where L is in metres and T in seconds.
   a) If heat increases a 10m pendulum by 5cm, how much does the period change?
   b) If heat increases a pendulum's length by 0.5%, what is the relative change in the period?

4. During a movie scene a camera is moving closer to a window that consists of a square topped by a semicircle. If on your monitor screen the side of the square portion appears to be 12 cm long, with a possible error of 0.3 cm, what is the differential estimate of the corresponding error in the area of the window?

5. You are standing 50 metres from the base of a building whose height you have to estimate by measuring the angle of elevation. If your estimate must be correct within 2% and the observed angle is approximately 60°, how accurately must you measure the angle?

What questions do you have for your instructor?