Chapter 7: Linear approximations — **Section 3**

Newton's method

What you need to know already:	**What you can learn here:**
➤ How to used derivatives to construct the linearization of a function.	➤ An iterative method based on linearizations that can provide approximate solutions of difficult equations.

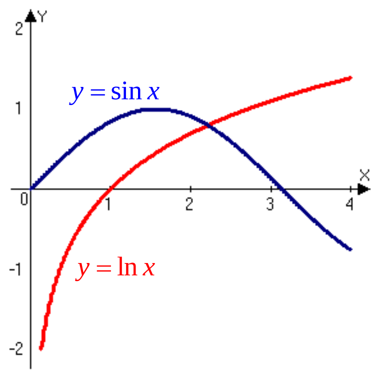
Can you find a solution of the equation $\ln x = \sin x$?

Let me get my calculator…

Newton did not have a calculator!

But I do!

I realize that, and we can use it to check if there actually is a solution, as this graph shows.



But we shall use this example to develop a method that can help us solve any difficult equation through an iterative process that…

A what?

An *iterative process* is one where we use the same formula over and over again to get a better and better estimate of the value we are looking for. You'll see how that works as we develop it.

This iterative process will allow us to get only an approximate value of the solution, but we can keep repeating the process as many times as we want, until we are satisfied with the precision of the estimate.

But why do we have to do that? We still have calculators!

The important aspect of this method is not its efficiency, which is low, but its clever use of derivatives and tangent lines to solve an important problem. Also, historically, this is the first of many methods that now allow our calculators to be the great computing machines they are.

Like using the linearization to estimate a function?

Exactly! So, focus on the ideas used in the method, since you will never use this method for practical purposes, but understanding and imitating its steps will help you better understand the whole point of calculus.

So, did Newton invent this method too?

Not really! He gave a very rudimentary idea that could only be applied to polynomial equations in a very complicated way. Other mathematicians kept making it simpler and more general, including a certain Joseph Raphson, whose name is often attached to that of Newton when referring to this method. Eventually, Thomas Simpson refined the method and brought it to the form that you will see here. By the way, we shall meet Simpson again when we shall study integrals.

So, going back to our mysterious equation, the first thing to notice is that, as with any other equation, we can move all its terms to the left side of the equal sign, thus writing it as:

$$\ln x - \sin x = 0$$

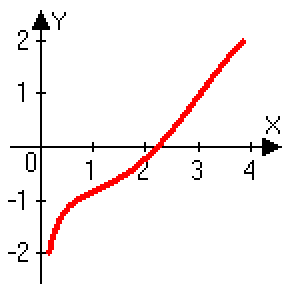
Knot on your finger

Any equation in one variable, say *x*, may be written in the form:

$$f(x)=0$$

Therefore, ***solving any equation*** in one variable is equivalent to ***finding the x-intercepts*** of a suitable function.

In our case $f(x)=\ln x-\sin x$ and if we look at the graph of this function, shown here, we see that its intercept seems close to 2. So we start our iterative process at the value $x=2$.

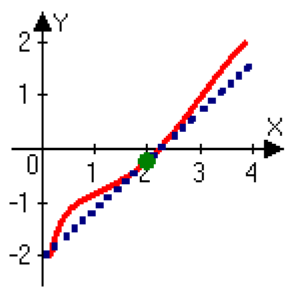


The reason why this method is in this chapter on linear approximation methods is that, just as we have done in the previous sections, we are going to use the tangent line to reach our goal.

We know that the solution we are looking for is near $x=2$, so we construct the tangent line to this function at $x=2$ and hope that its intercept is close to the one we are seeking, as the picture indicates.

The point of contact is $(2,\ln 2-\sin 2)$ and the slope of the tangent line is:

$$f'(x)=\frac{1}{x}-\cos(x)\quad\Rightarrow\quad f'(2)=\frac{1}{2}-\cos 2$$



Therefore its equation is:

$$y=\left(\frac{1}{2}-\cos 2\right)(x-2)+\ln 2-\sin 2$$

Well, here we see our function and its tangent line.

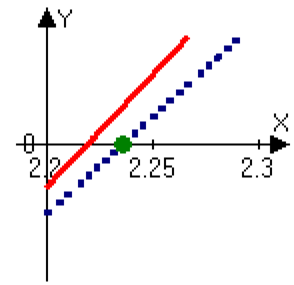
Actually, it looks like they have the same intercept!

That is because the tangent line is a good approximation to the curve, as long as we are close to the point of contact. However it is not the same. If we look closer we can see the difference.

At this point we can identify the *x* intercept of the tangent line:

$$0=\left(\frac{1}{2}-\cos 2\right)(x-2)+\ln 2-\sin 2$$

$$\Rightarrow\quad x=2-\frac{\ln 2-\sin 2}{.5-\cos 2}\approx 2.236$$



This does not give us the solution we seek, but a new and better estimate, so we can repeat the process starting from this new value.

By retracing our steps with a more general notation, we can come up with a formula that can be used for all later steps. In fact, let's get a formula that can be used for all similar problems!

We started with a reasonable estimate of the intercept. Let us call it x_0, since it is our starting point. Then we constructed the tangent line there:

$$y=f'(x_0)(x-x_0)+y_0$$

And finally, we computed the *x* intercept of this tangent line, which became our new estimate:

$$0=f'(x_0)(x-x_0)+f(x_0)\quad\Rightarrow\quad x=x_0-\frac{f(x_0)}{f'(x_0)}$$

We call this new estimate x_1 and are now ready to repeat the process with the new starting value x_1, so that our second, even better estimate will be:

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

And now that we have found the key to the process, there is no stopping us! Once we have the *n*-th estimate x_n we can compute the next one:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

Well, this is the general formula for Newton's method!

Strategy for Newton's method

To obtain an ***approximate solution*** of an equation:

1. Move all terms to the left, so that the equation becomes of the form $f(x)=0$
2. Construct the ***formula*** $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$
3. Select a reasonable ***starting estimate*** x_0
4. ***Apply*** the formula to get the next, better estimate x_1
5. ***Repeat*** the process until the estimate is stable or close enough.

Example: $\ln x=\sin x$

Let us retrace our steps through the strategy of Newton's method.

We moved all terms to the left:

$$\ln x-\sin x=0$$

We constructed the iterative formula:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\ln x_n-\sin x_n}{\frac{1}{x_n}-\cos x_n}$$

We selected the starting value $x=2$, then applied the formula once:

$$x_1=2-\frac{\ln 2-\sin 2}{.5-\cos 2}\approx 2.236$$

But we can now repeat the process and get an even better estimate:

$$x_2=2.236-\frac{\ln 2.236-\sin 2.236}{\frac{1}{2.236}-\cos 2.236}\approx 2.2192$$

$$x_3=2.2192-\frac{\ln 2.2192-\sin 2.2192}{\frac{1}{2.2192}-\cos 2.2192}\approx 2.2191$$

$$x_4=2.2191-\frac{\ln 2.2191-\sin 2.2191}{\frac{1}{2.2191}-\cos 2.2191}\approx 2.2191$$

We are not improving any more, at least not with the significant digits we are using, so we can stop here.

Notice that the formula that provides the next estimate of the solution is a function of the previous estimate. That means that if we enter this formula in a calculator as a function, we can implement the number-crunching steps more quickly.

Isn't that cheating? Weren't we supposed to pretend we did not have a calculator?

Remember that the key aspect of the method is in its use of tangent lines. In Newton's times people would have lots of time, no distractions –such as TV and computers – and a desire to explore the subject more. We are now going through the steps mostly to see how they work, not to actually implement the method for real.

Example: $\sin x=e^{-x}$

We move all to the left:

$$\sin x-e^{-x}=0$$

Construct the formula:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\sin x_n-e^{-x_n}}{\cos x_n+e^{-x_n}}=N(x_n)$$

Here I am using the notation $N(x_n)$ to indicate Newton's formula applied to the current estimate.

Where do we start? Let's use a simple value, such as $x = 0$. It's not a great estimate, but $\sin 0 = 0$ and $e^0 = 1$ are not that far away. In this way we get:

$$x_1 = N(0) = 0 - \frac{\sin 0 - e^{-0}}{\cos 0 + e^{-0}} = \frac{1}{2} = 0.5$$

We better use the calculator from now on:

$$x_2 = N(0.5) \approx 0.58564$$

$$x_3 = N(0.58564) \approx 0.588529$$

$$x_3 = N(0.588529) \approx 0.58853$$

Maybe we can stop here, since the improvement is far in the decimal digit, or we can continue if we so need or want.

It seems fairly simple, but I have a few questions.

And I am glad you do. What's the first?

Does this method work every time?

No! Sometimes the tangent line will take us further away from the solution, instead of closer, especially if the choice of starting estimate was not a good one. In that case you can either start from a different first guess, or investigate if the method is not applicable in that situation.

And how do we investigate that?

By using deeper properties of the method, properties that for now we shall leave alone. Hopefully you will come back to this method and study it in more detail later. For now, time to become familiar with the iterative formula by applying it in a few cases.

Summary

- Newton's method is an iterative method to obtain approximate solutions of difficult equations.
- Iterative means that the same formula is used over and over to obtain better and better estimates.

Common errors to avoid

- Don't expect Newton's method to be used in practical situations, as it is an obsolete method! Its importance is in the ingenious way to use tangent lines to obtain approximations.
- Don't expect Newton's method to work every time. Depending on the equation and the location of the initial estimate it may work slowly, take you to a solution different from the one you want, or not take you there at all!

Learning questions for Section D 7-3

Review questions:

1. Explain the geometrical basis for Newton's method.
2. Describe how to implement Newton's method.
3. Explain how the iterative formula of Newton's method is obtained.

Memory questions:

1. What is the recursion formula for Newton's method?
2. What is the purpose of Newton's method?

Computation questions:

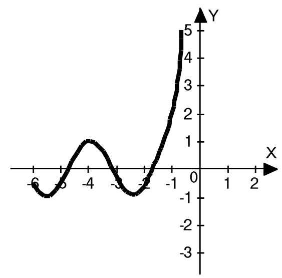
For each of the equations provided in questions 1-8:

- determine a suitable starting estimate
- construct Newton's iterative formula
- apply the formula three times, with the assistance of your calculator, to obtain an approximate value for a solution
- compare the value you found with the value provided by the calculator's "*zero*" or "*intersect*" function.

1. $x^4 + 4 = 6x^2$.
2. $6x^4 + 3x^2 - 10 = 2x^2 + 30$
3. $\sinh x + 3x = e^{-x}$
4. $\cosh x + 3x^2 = e^{-x}$
5. $3 = xe^x$
6. $\ln x + 3x = 2$
7. $e^{-x} + 3x = 2$
8. $e^x + 3x = 2$

9. Use Newton's method to estimate the smallest positive x intercept of the function $f(x) = x^4 + x\cos x - 53$ with 3 correct decimal digits.

10. A section of the graph of the parametric curve $\left(-e^t, \frac{1}{e^{3e^t}} + \sin 2e^t\right)$ is shown here. Construct Newton's iterative formula needed to estimate the *x* intercept closest to the origin and identify a suitable starting value, BUT all in terms of the parameter *t*.



Theory questions:

1. Why is Newton's method considered one type of linear approximation?
2. For what values of the second derivative will Newton's method work better?
3. How many steps of Newton's method would be needed if we were crazy enough to apply it to a linear equation?
4. If $x = c$ is a solution of the equation $f(x) = 0$ and $g(x) = x - \frac{f(x)}{f'(x)}$, for what function $k(x)$ is $x = c$ a solution of the equation $g(x) = k(x)$?

Application questions:

1. Assume that a physical quantity, denoted by *r*, is linked to another quantity, denoted by *p* through the function $r = p \ln p + \frac{p^2}{2} - 1$. Construct Newton's formula for this function and use it with 3 iterations to estimate the value of *p* which would make $r = 0$.

Templated questions:

1. Construct a reasonably simple equation that cannot be solved with usual algebraic methods, and then apply Newton's method to it to obtain an approximate solution.
2. Whenever you apply Newton's method for a few iterations, use the sequence of values you obtain to determine if the method is working or not. Always identify the reason for your conclusion.

What questions do you have for your instructor?