Chapter 8: Graphical analysis **Section 1**

Extreme points

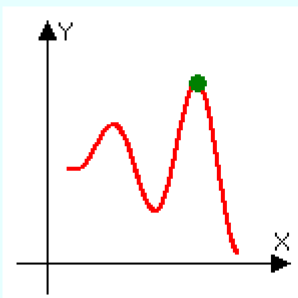
What you need to know already:	What you can learn here:
➤ How to solve basic algebraic and trigonometric equations. ➤ All basic techniques of differentiation and the graphical meaning of a derivative.	➤ How to use derivatives to identify the extreme values, that is maxima and minima, of the graph of a function.

The derivative was designed to provide the slope of the graph of a function, so it is not surprising that it can also provide further useful information about such graph. It is now time to make full use of this connection and to develop ways to analyze and identify features of the graph of a function by using derivatives.

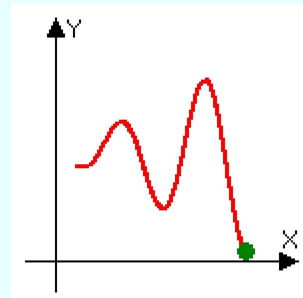
As it is so often the case in mathematics, it will be useful to begin by following Aristotle's advice and define some concepts precisely, or at least as precisely as we need them to be.

Definition

An ***absolute maximum*** for a function $f(x)$ is a ***point*** $(c, f(c))$ on the graph of $f(x)$ that is not lower than any other point of such graph.



An ***absolute minimum*** for a function $f(x)$ is a ***point*** $(c, f(c))$ on the graph of $f(x)$ that is not higher than any other point of such graph.



In simple words, an absolute maximum is the highest point and an absolute minimum is the lowest point on the whole graph, except possibly for other points with the same *y*-coordinate.

Before anyone starts objecting to these definitions, let me state that many authors refer to the ***number*** $f(c)$ as the maximum or minimum of a function, not the point. Such a choice makes sense from many points of view and I respect it highly. However, for a number of reasons related to my desire to explain these concepts in a simple way, I prefer to give the name of maximum or minimum to the ***point***, not just its *y* coordinate.

The same applies to the next definitions.

Definition

If the point $(c, f(c))$ is an absolute maximum or minimum of $f(x)$, then we call it an ***extreme point*** or ***extremum*** for $f(x)$.

In mathematics, whenever an object is defined we ask three basic questions:

1) Under which conditions can we be sure that it exists? (***Existence***)
2) Is there only one such object in any given situation, or can there be several? (***Uniqueness***)
3) How do we find one? (***Construction***)

In the case of extreme points, it is clear, by considering the sine and cosine functions, that they are not unique, so do not be surprised if you find more than one extreme point for the same function. What about the other two questions?

The existence question has a nice answer in the following classic theorem.

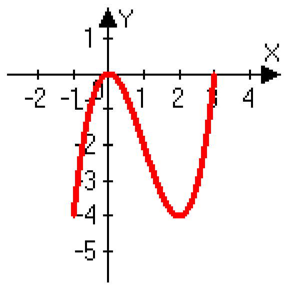
Technical fact

The "Extreme Value Theorem"

If a function $f(x)$ is ***continuous*** on a ***closed*** interval $[a, b]$, then it has at least one ***absolute maximum*** and at least one ***absolute minimum*** on $[a, b]$.

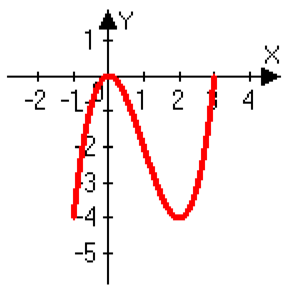
The formal proof of this theorem is rather technical and I will skip it. If you are interested in seeing one, you can start here. However, it is an important theorem since it can be used both in later calculus theory and in applied problems.

Example: $y = x^3 - 3x^2, -1 \le x \le 3$



This is a polynomial function, so it is continuous. Since we are restricting its domain to the closed interval $[-1, 3]$, the extreme value theorem assures us that it has at least one absolute maximum and one absolute minimum. The graph, shown here, confirms this conclusion, although we are still not sure of where these points occur. It looks like they occur at $x = -1, 0, 2, 3$, but we cannot be sure based on the graph only.
We need to wait until we see the answer to the construction problem.

Example: $y = x^3 - 3x^2, -1 < x < 3$



This is the same function as before and its graph looks exactly the same on the calculator, BUT…

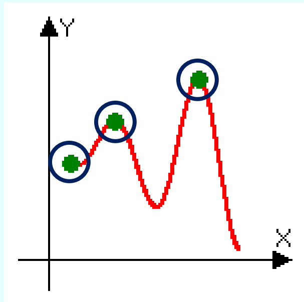
But the end points of the domain are NOT included and therefore the *Extreme Value Theorem* does not apply and does not guarantee the existence of extreme values.

Notice, however, that this function still has an absolute maximum, at or near $x = 0$, and an absolute minimum, at or near $x = 2$. The fact that we cannot use the theorem does not mean that the function has no absolute extrema: it is a one-way only theorem!

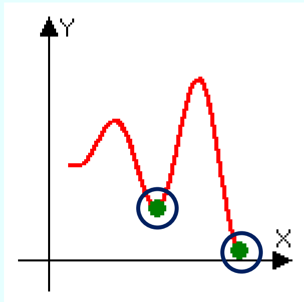
This last example shows a weakness of existence theorems: they do not tell us how to find what we are looking for, nor can they assure us of the absence of the feature. So we need to look at how to solve the construction problem. Before doing that, we need a few more definitions.

Definition

A ***relative*** or ***local maximum*** for a function $f(x)$ is a ***point*** $(c, f(c))$ on the graph of $f(x)$ that is not lower than any other point of such graph on some x interval (a,b) around c.



A ***relative*** or ***local minimum*** for a function $f(x)$ is a ***point*** $(c, f(c))$ on the graph of $f(x)$ that is not higher than any other point of such graph on some x interval (a,b) around c.



Local maxima and minima are collectively called ***local extrema***.

In simple words, a relative or local extremum is the highest or lowest point for a section of the graph that does not include end points

Lots of definitions! I am glad that I have seen them before!

I agree, and I hope you have seen the next two as well: I include all these definitions for your convenience and for completeness. By the way, you will find that the next two definitions are also presented a little differently by some authors. Again, my choice is based on my desire to keep things simple.

Definition

A ***cut point*** of a function $y = f(x)$ is a value $x = c$ such that either:

- $f(c) = 0$ or
- $f(c) = DNE$ and c is at the boundary of the domain of $f(x)$

Example: $f(x) = \dfrac{x^2 - 4}{\sqrt{x^2 - 1}}$

The cut points of this function are $x = \pm 2$, because the function becomes 0 there, and $x = \pm 1$, because the function is undefined there AND these are at the boundary of the domain of the function, since such domain is:

$$D = (-\infty, -1) \cup (1, \infty)$$

Notice that the function is also undefined at $x = 0$, or anywhere between -1 and 1, but these are not cut points as they are totally outside the domain.

Definition

A ***critical value*** of a function $f(x)$ is a cut point of $f'(x)$ that is in the domain of $f(x)$.

Example: $f(x)=\dfrac{x^2+4}{x-1}$

The derivative of this function is:

$$f'(x)=\frac{2x(x-1)-(x^2+4)}{(x-1)^2}=\frac{x^2-2x-4}{(x-1)^2}$$

with cut points at $x=1\pm\sqrt{5}$ (derivative becomes 0) and $x=1$ (derivative is undefined). However, only $x=1\pm\sqrt{5}$ are critical values, since they are the only ones in the domain of $f(x)$.

And now, drum roll please!

Technical fact

The extended "Fermat's theorem"

If a function $f(x)$ has a ***relative extremum*** at $(c, f(c))$, ***then*** c is a ***critical value*** of $f(x)$.

Proof

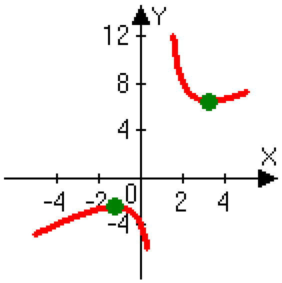
Again the proof is rather technical and not useful for our goals and you can find it here.

However, to convince yourself of its truth, just consider the fact that if $f'(c)>0$, the function's slope is positive, hence the function is going up, while if $f'(c)<0$, the slope is negative and the function is going down. But at a relative extremum the function is neither going up nor down! Therefore, we are left with the other two options: either $f'(x)=0$ or it is undefined.

Keep in mind that this is not a formal proof, just a simple argument that I hope can convince you.

Example: $f(x)=\dfrac{x^2+4}{x-1}$

We just saw that this function has critical values at $x=1\pm\sqrt{5}$. If we look at the graph of this function, we notice that at $x=1-\sqrt{5}$ there is a relative maximum and at $x=1+\sqrt{5}$ a relative minimum.



Notice that neither is an absolute extremum, but that no other relative extrema exist for this function.

What does the other cut point of the function correspond to?

Why did you call it the "extended" Fermat's theorem?

Because Fermat only looked at the case where the function is differentiable, thus excluding the case when the derivative is undefined. But the statement works more generally in the form I stated, considering the definitions it relies on..

Incidentally, Fermat proposed this theorem without knowing what a derivative was, since he worked on it well before Newton and Leibniz invented derivatives! He used a different approach that only worked for differentiable functions, but the result is the same. In fact, Fermat invented the Cartesian plane before Descartes, but he was a lawyer who did math for fun and did not publish his discoveries. What a genius, eh? In fact, he is often considered one of the fathers of calculus.

Back to the theorem, it also gives us a great strategy to look for extreme points.

Strategy for finding extreme points of a function

To identify all extreme points of a function $f(x)$:

1. Determine all ***critical values*** of $f(x)$, including the end points of its domain.
2. ***Check*** whether each of them is a relative minimum, a relative maximum, or neither.
3. Among all relative extrema, ***choose*** the absolute ones by picking the highest and lowest points.

What do you mean by "neither"? Doesn't it have to be one or the other?

No! The theorem states that *if* there is an extremum, it occurs at a critical value, but it does not claim that every critical value corresponds to an extremum. Some critical values are neither!

Example: $y = x^3 - 3x^2, -1 \le x \le 3$

The derivative of this function is:

$$y' = 3x^2 - 6x = 3x(x-2), \ -1 < x < 3$$

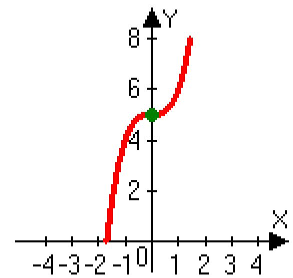
Notice that the derivative is not defined at the end points, since the function is only continuous on one side.

Therefore, the critical values are at $x = 0, 2$, where the derivative is 0, and at $x = -1, 3$ where the derivative is undefined, but the function exists.

These are exactly the values we found earlier by looking at the graph.

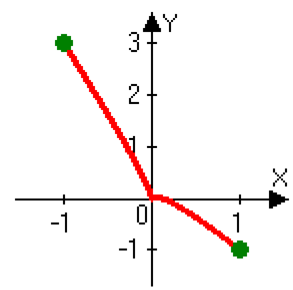
Example: $y = x^3 + 5$

The derivative of this function is $y' = 3x^2$, so that its only critical value is at $x = 0$. But as we can see from the graph, the function has neither a maximum nor a minimum there.



Example: $y = x^{2/3} - 2x, -1 \le x \le 1$

This function is continuous and its domain is a closed interval, so, by the EVT, it has both absolute maximum and minimum. But where are they, and does the function have other relative extrema? Here is a first graph:

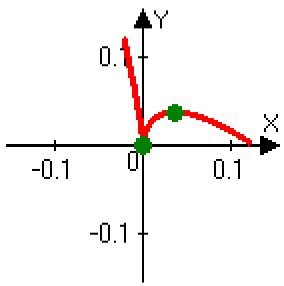


It looks like the end points are the only extrema, but let us use Fermat's theorem to confirm this. The derivative of this function is:

$$y' = \frac{2}{3}x^{-1/3} - 2 = 2\frac{1-3x^{1/3}}{3x^{1/3}}$$

This is 0 at $x = \frac{1}{27}$ and undefined at $x = 0$ and at the end points $x = \pm 1$.

These are all critical values, since they are all in the domain of the original function. Therefore, any extrema must be at some of these points, but which ones and are they all extrema? By looking at the previous graph, we can see that there is an absolute maximum at $x = -1$ and an absolute minimum at $x = 1$, but what about the other two? By using a more focused window we can see them more clearly:



We have a relative minimum at $x = 0$, a relative maximum at $x = \frac{1}{27}$.

Notice that in this example the relative minimum occurs for a value of x for which the derivative is undefined, NOT equal to 0.

Do we have to rely always on the calculator's graph to check what kind of extreme point we have at a critical value?

Certainly not, since the first developers of calculus did not have a calculator and still could figure it out!

I suspect that you either know or have a pretty good idea of how to classify critical values, that is, how to decide whether they provide a maximum, minimum or neither. But we'll discuss a proper strategy to do that in the next section.

Summary

- An extreme point is one where the function reaches a maximum or a minimum, either locally or globally.
- The *Extreme Value Theorem* tells us that every function that is continuous over a closed interval reaches at least one absolute maximum and at least one absolute minimum.
- *Fermat's theorem*, in its extended version, tells us that relative extrema are only found at the critical values, that is, at values in the domain of the function where the derivative is either 0 or undefined. This generates a simple strategy to identify all extreme points of a function.

Common errors to avoid

- Do not assume that every critical value corresponds to a maximum or a minimum: it may be neither!
- Don't ignore the end points of the domain, if they are included in the domain, since they are critical values too.

Learning questions for Section D 8-1

Review questions:

1. Explain what the *Extreme Value Theorem* states.
2. Explain what *Fermat's Theorem* states.
3. Describe the difference between the *Extreme Value Theorem* and *Fermat's Theorem*, but don't just restate them!
4. Describe the strategy for identifying all possible extreme points of a function.

Memory questions:

1. Which theorem guarantees the existence of extreme points?
2. Which theorem is used to find the location of the extreme points?
3. What is the derivative of a function at one of its maximum or minimum values?
4. What are the two conditions under which we can use the *Extreme Value Theorem*?
5. Which two conditions define a critical value?

Computation questions:

For each of the functions in questions 1-20, identify and classify all extreme values, that is:

a) Use the *Extreme Value Theorem* to determine if extreme points exist.
b) Use *Fermat's Theorem* to identify which points can possibly be maximum or minimum points.
c) Use the calculator's graph to determine if each of them is a maximum, a minimum or neither.

1. $y=1+\left(x^2-2\right)^3$ on $[-2,1]$.
2. $y=x^3-3x+1$
3. $y=2x^3-12x^2+18x+30$
4. $y=6x^2-x^3$
5. $y=\frac{1}{4}x^4+\frac{2}{3}x^3-12x^2+10$
6. $y=x^4-12x^2$

7. $y = \dfrac{1}{x^2}$ on $1 \le x \le 3$.

8. $y = \dfrac{1}{x^2}$ on $-1 \le x \le 1$.

9. $y = \dfrac{5x}{x^2+3}$ on (-1, 1)

10. $y = \dfrac{5x}{x^2+3}$ on [-1, 1]

11. $y = \dfrac{5x}{x^2+3}$

12. $y = \dfrac{5x}{x^2+3}$ on $(0, \infty)$

13. $y = x + 2\sin x$ on [-3, 3]

14. $g(x) = 2x + \cos 2x$ on $[0, 2\pi]$

15. $f(x) = \dfrac{2\cos x}{2+\sin x}$ on $[0, 2\pi]$

16. $y = x^2 - \tan x^2$

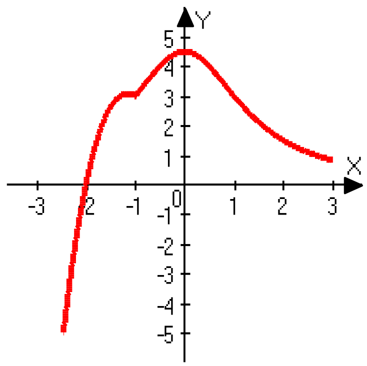
17. $y = x^2 e^x$ on [-3, 1]

18. $y = e^x - e^{2x}$ on [-1, 1].

19. $y = \dfrac{x}{\ln x}$ on [1, 4].

20. $y = x^2 - \ln x^3$ on $\left[\dfrac{1}{2}, 2\right]$.

21. The function $y = \begin{cases} x^3 - 4x & \text{if } x \le -1 \\ \dfrac{9}{x^2+2} & x > -1 \end{cases}$ has the graph shown here.



Use calculus to identify all possible critical values of the function and use the graph to classify each such point as a maximum, minimum or neither.

22. For what values of *a* and *b*, if any, does the function $y = ax^2e^{bx}$ have a minimum at $(1, 2)$?

23. The function $f(x) = x^2 + e^{ax} + \dfrac{b}{x+1}$ has a local minimum at $(0, 2)$. What can *a* and *b* be?

24. A function has formula $f(x) = ax^3 + bx^2 + x$ and has critical values at $x = 1, 2$. Determine the values of *a* and *b* and classify these two critical values.

Theory questions:

1. If $f\,'(3)=0$, does that mean that at $x=3$ there is a maximum or a minimum?

2. Is it possible that a function $y=f\left(x\right)$ has a maximum at a point where the derivative is not 0?

3. Which values are cut points for a derivative, but are not critical values?

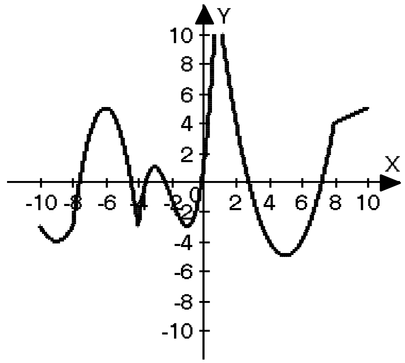
4. Any cubic function is continuous, but has no absolute maximum. Why doesn't that contradict the Extreme Value Theorem?

5. Is it possible for a function to have several critical values, but no maxima or minima?

6. How do we distinguish between extreme points and vertical asymptotes?

7. Why is the restriction of a function to a closed interval useful in applications?

8. Use the graph shown here to answer the following questions about the function it represents:



a) Does the function show the presence of local maximum and minimum points?
b) Does the Extreme Value Theorem apply to this function?
c) Where do its relative maxima and minima seem to occur?
d) What can we say about the derivative at each relative maximum or minimum?
e) Does Fermat's theorem apply to this function?

Proof questions:

1. Decide which functions in the family described by the formula $y=x^2\sqrt{x^2-c^2}\,,\,c>0$ have extreme points, if any, and if so, of what type.

Templated questions:

1. Identify and classify the extreme points of the functions listed in the document Sample functions to analyze.

2. Construct a reasonably simple function and use the methods of this section to identify and classify its extreme points.

What questions do you have for your instructor?